Lagrangian Turbulence, Generalized Flows, and Irreversibility

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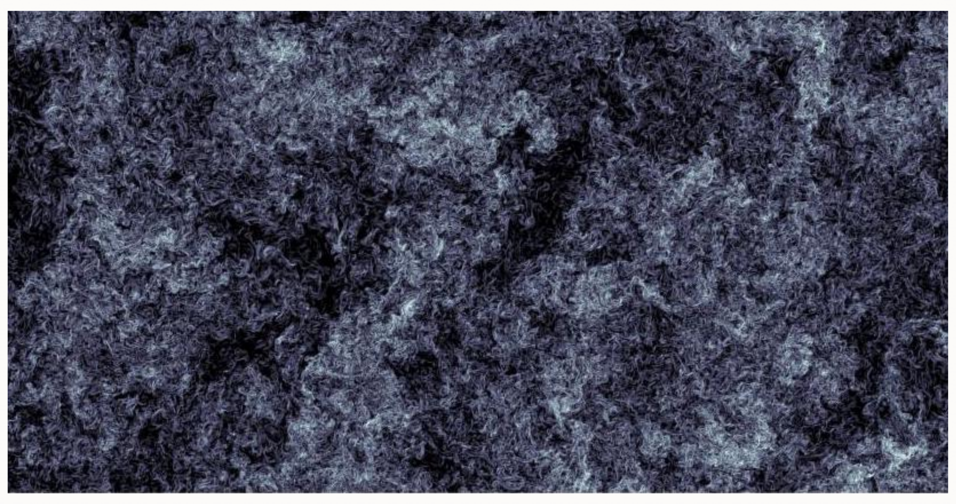
An idea underlying most turbulence models is to describe the large scales by Euler inviscid equations. The concept of solution must then be weakened to obtain turbulent fields that are sufficiently rough to provide a finite dissipation of kinetic energy. This strongly questions classical approaches because solutions become spontaneously stochastic and non-unique. Still, this opens the way to new strategies able to provide an intrinsically probabilistic construction of solutions.



Local energy dissipation in a slice of a three-dimensional homogeneous turbulent flow at $R_\lambda \approx 730$.

Turbulence vs. Euler inviscid dynamics

$$\nabla \cdot \boldsymbol{u} = 0$$
$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \qquad \nu \to 0?$$

- Dissipative anomaly: $\varepsilon_{\mathrm{D}} = \frac{\nu}{2}\left\langle \|\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}\|^2 \right\rangle \to \text{const}$
- Onsager's criterion: $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\ell}) - \boldsymbol{u}(\boldsymbol{x})| \sim \ell^h$ with $h \le 1/3$

⇒ Turbulent velocity fields are rough

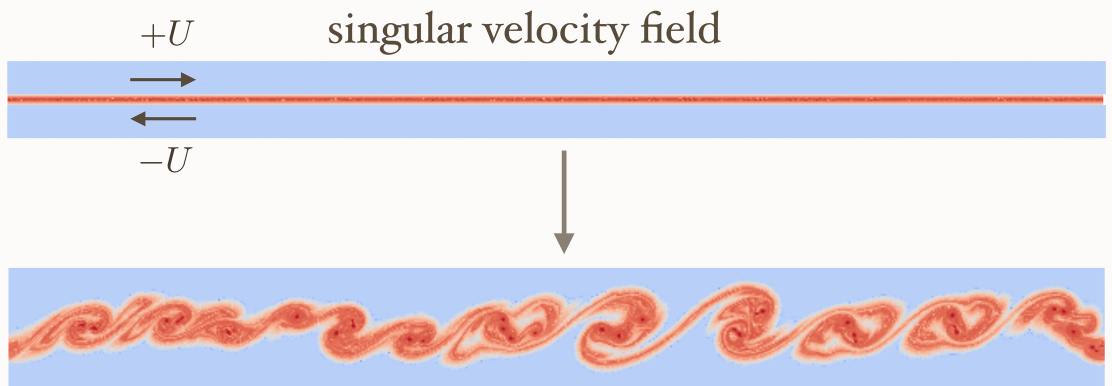
Construct admissible (dissipative) weak solutions to Euler's equation

⇒ Almost all non-smooth initial conditions give non-uniqueness

Brenier et al. (2011); Buckmaster et al. (2016)

Spontaneous stochastic behavior of rough velocities

Example: Kelvin–Helmholtz vortex sheet



The mixing layer reaches a finite size in a finite time, even when the initial perturbation is sent to zero

Inviscid dynamics with a singular initial condition is less predictable than any chaotic system

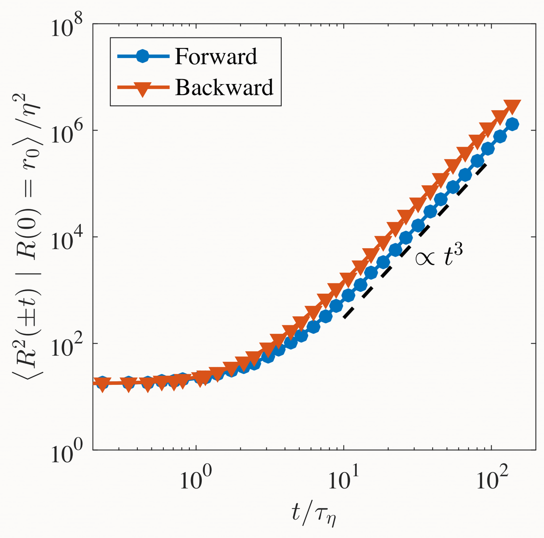
Tracers explosive separation

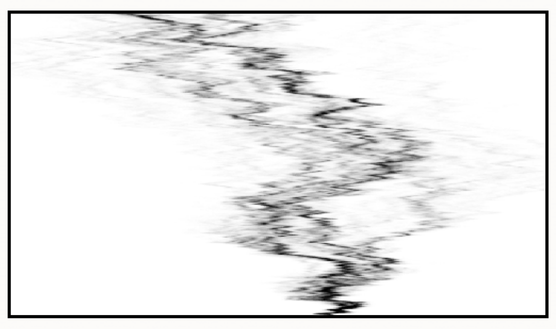
Richardson's law

$$\boldsymbol{R}(t) = \boldsymbol{X}(t, \boldsymbol{x}_0) + \boldsymbol{X}(t, \boldsymbol{x}_0')$$

$\frac{\mathrm{d}R}{\mathrm{d}t} \propto R^h$ non-Lipschitz

$R(t) \sim t^{1/(1-h)}$ Explosive separation





The Lagrangian flow is non-unique

This largely explains intermittency of advected passive scalars

Falkovich et al. (2001)

Need for an intrinsically probabilistic description

- Weak solutions to Euler equation are very often non-unique
- The velocity can be spontaneously stochastic
- Non-differentiable velocities lead to explosive separation of tracers
- A non-unique Lagrangian flow explains anomalies of passive scalars

⇒ Suggests to relax the notion of "velocity field"

$\boldsymbol{x}$ → $p(\boldsymbol{x}', t+\delta t \,|\, \boldsymbol{x}, t)$ $\iff$ $\boldsymbol{u}(\boldsymbol{x}, t) \longrightarrow \gamma_{(\boldsymbol{x},t)}(\mathrm{d}\boldsymbol{u})$ Young measure

DiPerna and Majda (1987): distributional solutions

Lagrangian formulation of inviscid dynamics

$$\partial_{tt} \boldsymbol{X}(t, \boldsymbol{x}_0) = -\nabla p(\boldsymbol{X}(t, \boldsymbol{x}_0), t) + \sqrt{2\nu}\, \boldsymbol{\eta}(t)$$

Pressure ≡ Lagrange multiplier for incompressibility constraint

Arnold 1966 least-action principle:
Regular inviscid flow in a compact domain follows a geodesic on the manifold of volume-preserving maps.

Trajectories of fluid elements minimize kinetic energy:

$$\mathcal{A}_{0,t_{\mathrm{f}}}[\boldsymbol{X}(\cdot)] := \int_0^{t_{\mathrm{f}}} \mathrm{d}t \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}_0 \, \frac{1}{2} |\partial_t \boldsymbol{X}(t, \boldsymbol{x}_0)|^2 \longrightarrow \inf \qquad \text{subject to}$$

$$\boldsymbol{X}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0, \quad \boldsymbol{X}(t_{\mathrm{f}}, \boldsymbol{x}_0) = \boldsymbol{X}_{\mathrm{f}}(\boldsymbol{x}_0), \text{ and } \left| \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}_0} \right| \equiv 1$$

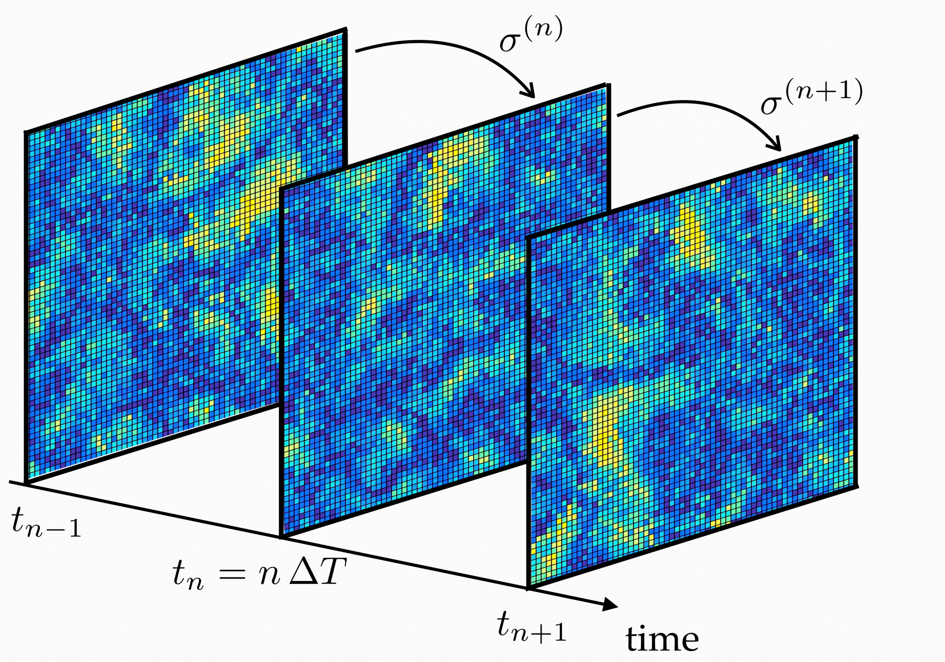
Brenier's generalized flows

Extend the least action principle to probability measures on paths $p[\boldsymbol{X}(\cdot,\cdot)]$

$$\langle \mathcal{A}_{0,t_{\mathrm{f}}}[\boldsymbol{X}(\cdot)] \rangle := \int \mathcal{D}\boldsymbol{X} \; p[\boldsymbol{X}(\cdot)] \, \mathcal{A}_{0,t_{\mathrm{f}}}[\boldsymbol{X}(\cdot)] \longrightarrow \inf$$

Kinetic energy is minimized in average

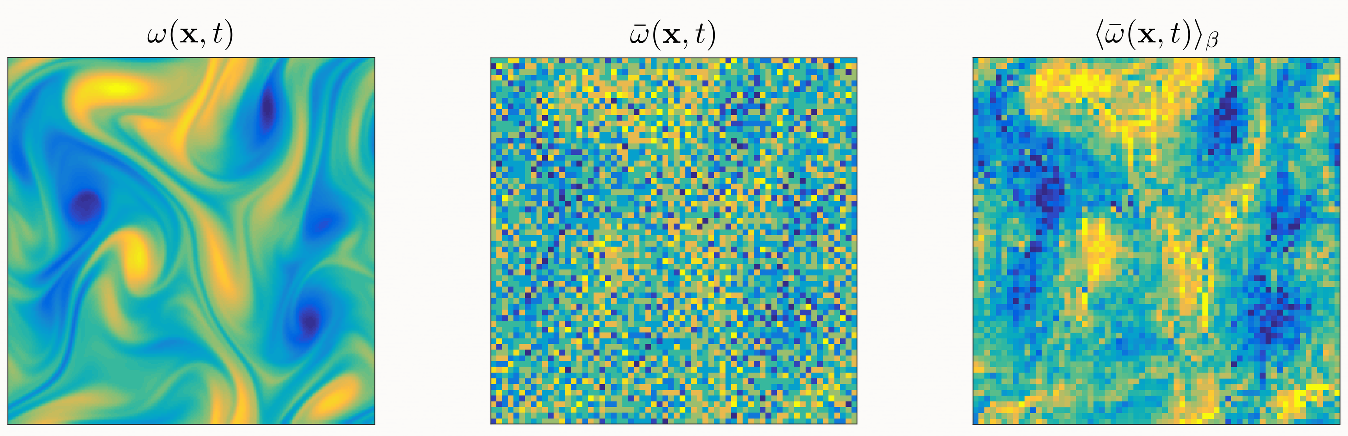
Brenier (1989)



👎 Artefact: can lead to a spurious probabilistic behavior when t_{f} is too large

👍 Allows constructing a space-time coarse-grained inviscid dynamics in terms of doubly-stochastic matrices (transition probabilities)

Application to the two-dimensional direct cascade



Generalized least-action principle is able to reproduce an irreversible dynamics, but this case has no anomaly and involves regular velocity fields

Perspectives

- Extensions to three-dimensional flows
- Exploit the least-action principle to derive generalized (Noether) invariants
- Bridge explosive separation of tracers and Eulerian spontaneous stochasticity

References

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