Seminaire Calisto, Inria December 14, 2021.

From Random trajectories to random fields :

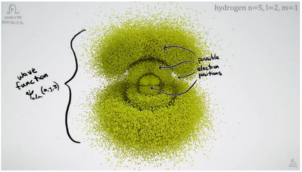
Is fluid dynamics intrinsically random ?

Simon Thalabard

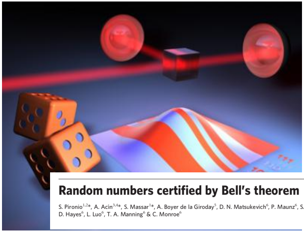
Layout

1. **Apparent vs intrinsic randomness**
2. The spontaneous stochasticity mechanism
3. Open projects

Quantum world: Randomness is intrinsic



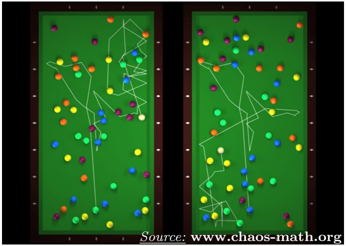
Source: www.minutephysics.com



Random numbers certified by Bell's theorem

S. Pironio[1,2]*, A. Acín[3,4]*, S. Massar[1]*, A. Boyer de la Giroday[5], D. N. Matsukevich[6], P. Maunz[6], S. Olmschenk[6], D. Hayes[6], L. Luo[6], T. A. Manning[6] & C. Monroe[6]

Classical world: Randomness is only apparent



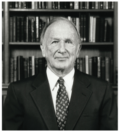
Source: www.chaos-math.org

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

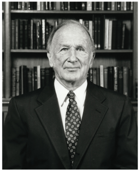


Lorenz '62 : Unpredictability ties to chaotic exponentiation of small initial errors:

$$\delta(t) = \delta_0 e^{\lambda t}$$

Lorenz '69 conjecture

The predictability of a flow which possesses many scales of motion



By EDWARD N. LORENZ, *Massachusetts Institute of Technology*[1]

(Manuscript received October 31, 1968, revised version December 13, 1968)

ABSTRACT

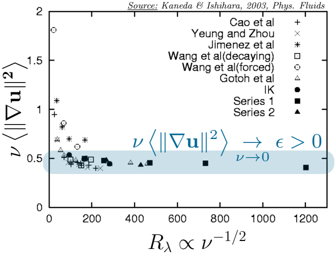
It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. The hypothesis is investigated with a simple mathematical model. An equation whose dependent variables are ensemble averages of the "error energy" in separate scales of motion is derived from the vorticity equation which

- **Formally deterministic** dynamics
- **Finite-time** emergence of randomness
- **Outcome independent from the observer**:
  unlike chaotic exponentiation, where finite-time errors can be made arbitrarily small

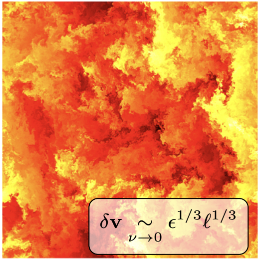
$\Rightarrow$ **Intrinsic, yet classical randomness.**

Random

Dissipative



Rough



Navier-Stokes equations

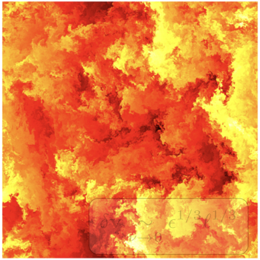
$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nu \nabla^2 \mathbf{v} + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0$$

Random

Dissipative

$$\nu \left\langle \|\nabla \mathbf{u}\|^2 \right\rangle \underset{\nu\to 0}{\to} \epsilon > 0$$

Rough



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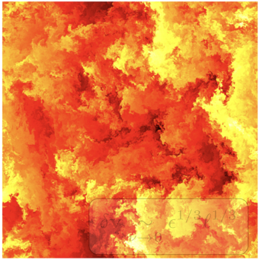
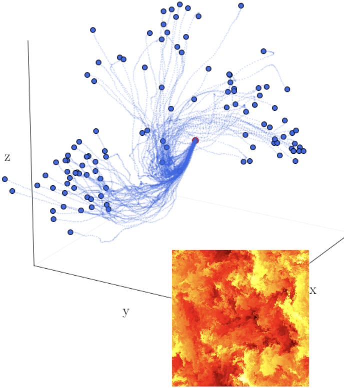


Two simpler examples of intrinsic randomness

Fluid trajectories



Shear layer instability

In both examples, intrinsic macroscopic randomness will emerge from a subtle interplay between **thermal noise** and the presence of some type of small-scale **roughness**.
This is the framework of **spontaneous stochasticity**.

1. Apparent vs intrinsic randomness
2. **The spontaneous stochasticity mechanism**
3. Open projects

Roughness produces infinite amplification of thermal noise in finite-time.

Random trajectories

SDEs

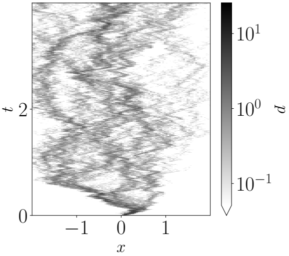
Toy models

ODEs

Random fields

PDEs

Roughness produces infinite amplification of thermal noise in finite-time.



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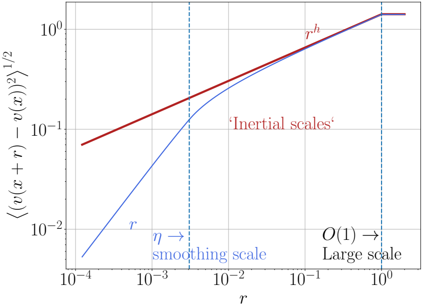
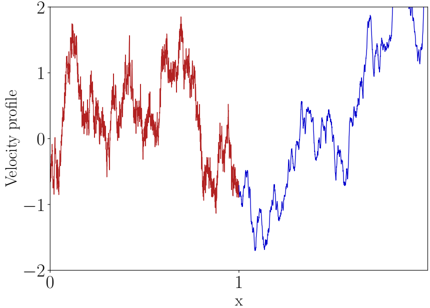
ADVECTION IN RANDOM GAUSSIAN FIELDS

$$\mathrm{d}\mathbf{X}_{\kappa,\eta} = \mathrm{d}v_\eta(t,\mathbf{X}) + \sqrt{2\kappa}\,\mathrm{d}\mathbf{W}, \qquad v_\eta(x+r) - v_\eta(x) \sim r^h \;\; as \;\; r,\eta \to 0$$

$h \in [0,1]$: spatial roughness η: smoothing κ : thermal noise.



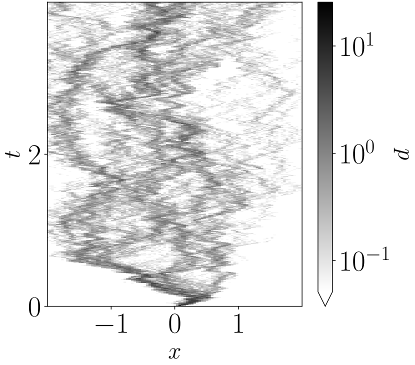


$$\langle dv_\eta(t,\mathbf{x})dv_\eta(t',\mathbf{x}')\rangle = C_{\eta,h}(\mathbf{x}-\mathbf{x}')\delta(t-t')dt$$

For suitable limits $\eta, \kappa \to 0$, initially coincident trajectories may reach O(1) separations,

- in finite time,
- with probability 1.

$\Longrightarrow$ **such trajectories are "spontaneously stochastic"**.



$$\textbf{Spontaneous stochasticity} \iff \exists \lim_{\substack{r_0,\eta\to 0\\ \kappa\to 0}} \mathbb{P}\left[\tau_1 < \infty\right] = 1,$$

with $\tau_\eta(r_0,\eta,\kappa) := \inf\{t, \|R\|_\eta < \eta\}$ $\qquad \tau_L(r_0,\eta,\kappa) := \inf\{t, \|R\|_\eta > L\}$

Example: 2 particles in a rough 1d field

- Separations are governed by the operator

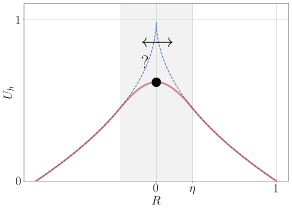
$$\mathcal{L}_2 := 2K_2\partial_{rr}, \qquad K_2 := \frac{D_0}{2}\|r\|_\eta^\xi + \kappa$$

- Whether or not particles separate depends on the small-scale!

$$\mathbb{P}\left[\tau_1 < \tau_\eta\right] = \frac{r_0 - \eta}{1-\eta} \to_{r_0,\eta\to 0} 0$$

- **Statistics map to Bessel process in dimension** $d_f = 2\dfrac{1-\xi}{2-\xi}$

Roughness produces infinite amplification of thermal noise in finite-time.



Random trajectories

SDEs

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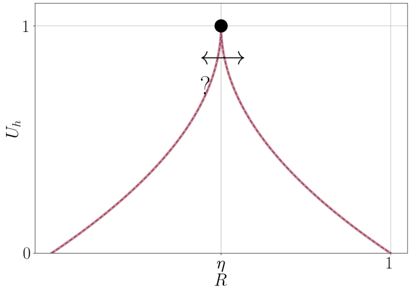
Overdamped Brownian particle in singular potential

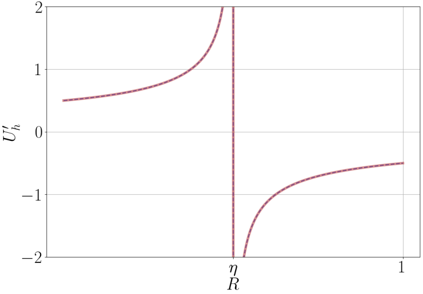
Algebraic separation $R(t) := X'(t|0) - X(t|0)$

$$\mathrm{d}R = -U'_{h,\eta}(R)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W, \qquad U_{h,\eta}(R) := 1 - \|R\|_{\eta}^{1+h},$$

$h < 1$: spatial roughness η: smoothing κ : thermal noise.

How much time to reach $|R| = 1$ from $|R| = 0$?





$h < 0$, $\eta = 0$

Overdamped Brownian particle in singular potential

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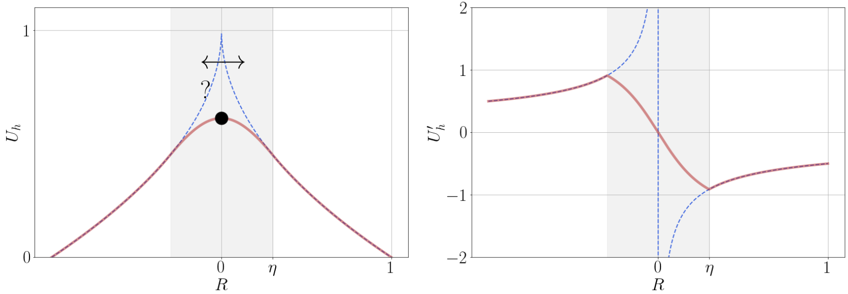
How much time to reach $|R| = 1$ **from** $|R| = 0$ **?**



$h < 0,\ \eta \neq 0$

Overdamped Brownian particle in singular potential

Algebraic separation $R(t) := X'(t|0) - X(t|0)$

$$\mathrm{d}R = -U'_{h,\eta}(R)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W, \quad U_{h,\eta}(R) := 1 - \|R\|_\eta^{1+h},$$

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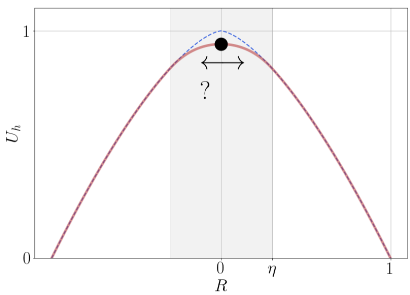
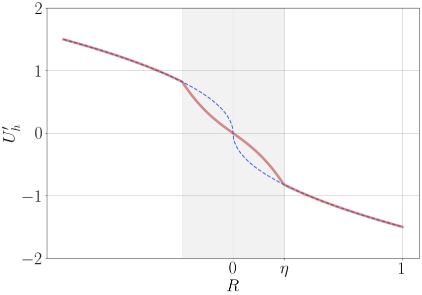
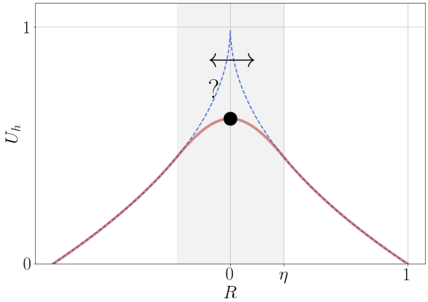




$0 < h < 1, \eta \neq 0$

$-1 < h < 1$

Average escape time in the limit $\eta, \kappa \to 0$



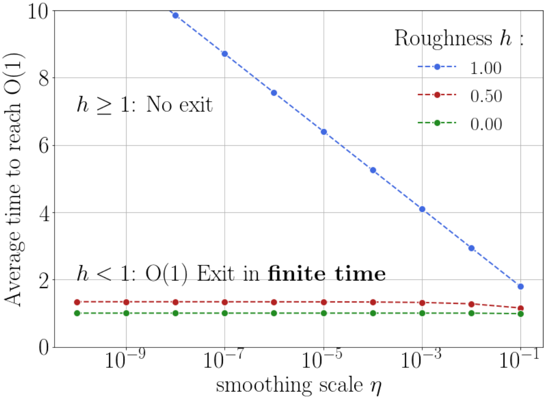
(i) From 0 to η

$$\tau(0 \to \eta) \propto O(\eta^{1-h}) + O(\eta^2/\kappa)$$

(ii) From η to 1

$$\tau(\eta \to 1) \propto \frac{1}{(1-h)(1+h)} = O(1)$$

This suggests the spontaneously stochastic scaling $\kappa \propto \eta^{1+h}$



The limit $\eta \to 0$ is spontaneously stochastic for $-1 < h < 1$:

- **Formally deterministic:** The amplitude of the noise vanishes $\kappa \to 0$
- **Remanently stochastic:** Particles starting from 0 reach O(1) separations in finite-time.

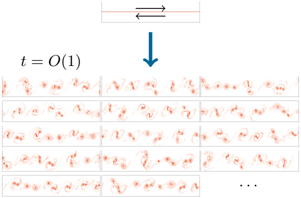
Roughness produces infinite amplification of thermal noise in finite-time.

Random trajectories

SDEs

Toy models

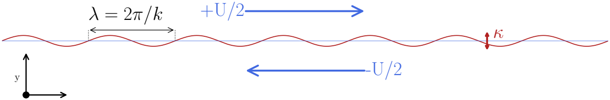
ODEs



Random fields

PDEs

Singular shear-layer instability



Linear inviscid theory:

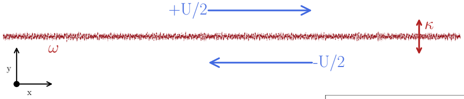
Exponential amplification with
growth rate $\sigma(k) = Uk/2$

When the perturbation scale vanishes, e.g. $k \to \infty$,
the growth rate explodes: $\sigma(k) \to \infty$

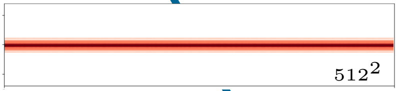
$\Rightarrow$ **Breakdown of linear theory**

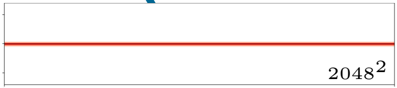
When the amplitude vanishes, the inviscid problem becomes ill-posed

$\Rightarrow$ **Singular initial-value problem**



Physical regularization by white noise + viscosity

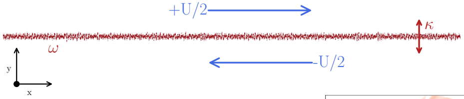




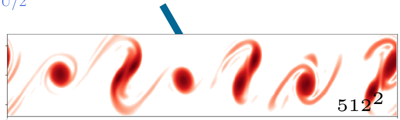
Fixed time

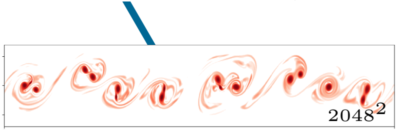
$t = 0$

$\nu, \kappa \to 0$



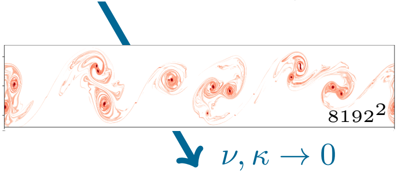
Physical regularization by white noise + viscosity





Fixed time

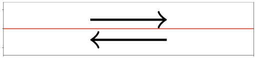
$t = 0.8$



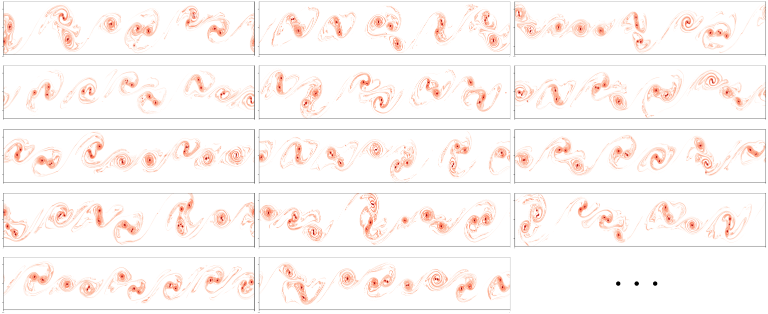
Numerical observations:

(i) Finite-time amplification

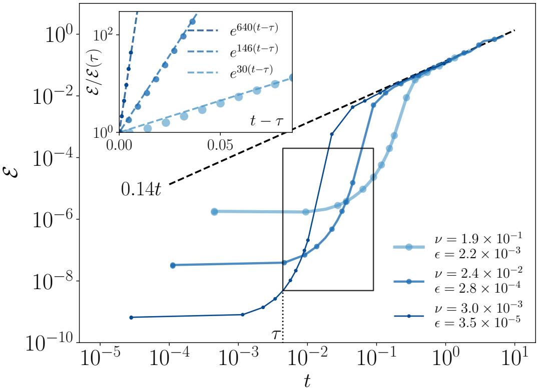
(ii) Infinite gain



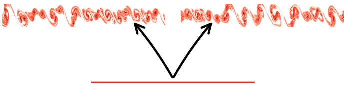
$$t = O(1)$$



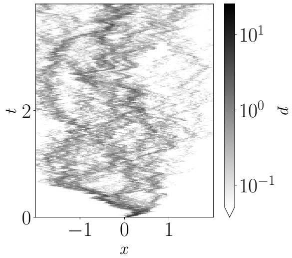
Example of measurement: **Explosive separation of velocity fields**



In the limit $\eta, \kappa \to 0$,
the dynamics is stochastic from $t = 0^{+}$,
but the underlying equations are formally deterministic.
$\Rightarrow$ **It is intrinsically random**

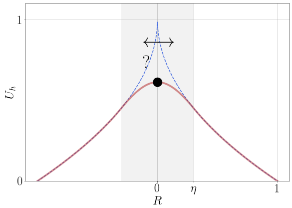


Roughness produces infinite amplification of thermal noise in finite-time.



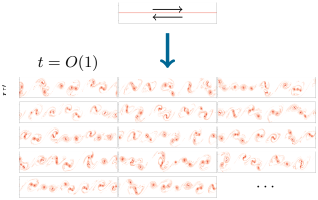
Random trajectories

SDEs



Toy models

ODEs



Random fields

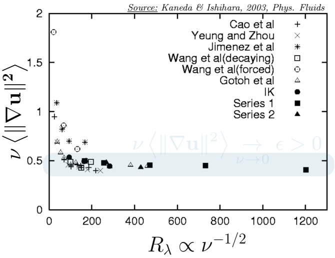
PDEs

1. Apparent vs intrinsic randomness

2. The spontaneous stochasticity mechanism

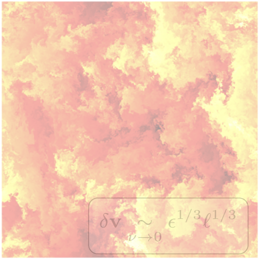
3. **Open projects**

Dissipative

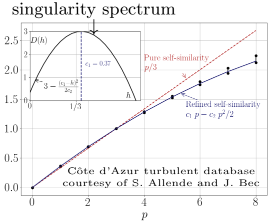


Random

Rough



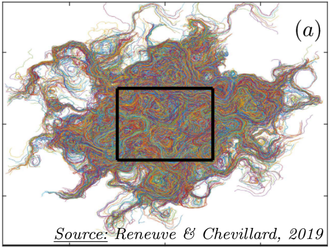
Multifractal



How does multi-fractality and irreversibility alter spontaneous stochastic motion of fluid particles?

Methods

Multi-fractal random fields introduced by Chevillard, based on Gaussian multiplicative chaos to model intermittency.



Road map

1. Numerical simulations of SDE
2. Qualitative behavior of trajectories
3. Connection to DNS signatures of Lagrangian irreversibility

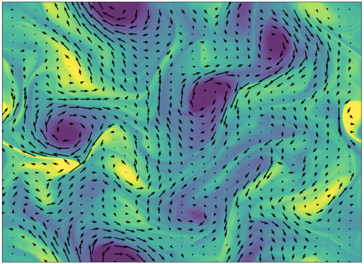
Possible developments

- Pink intermittency of scalar fields
- Statistical geometry of trajectories

Can one identify post blow-up signatures of stochasticy in SQG turbulence?

Methods

Numerical experiments and statistical analysis of SQG dynamics.



Road map

1. Pre-blowup: Scenario to singularities
2. Blow up : Fate of localized/homogenous perturbations
3. Fully developed: Lagrangian stochasticity vs intermittency

Possible developments

- Generalized SQG flows
- Stochasticity of 3D turbulence onset

Thank you for your attention!