# Self-similar blowups in cascading fluid models

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The blowup problem in mathematical fluid dynamics addresses the settlement of non-Lipschitz singularities within a finite-time horizon, out of sufficiently smooth initial data. One of the simplest scenarios is that of asymptotic self-similarity, which describes the finite-time algebraic convergence towards a universal self-similar profile, prescribed as a non-linear eigenvalue problem. Examples of self-similar blowups are found in the simplified setting of cascade models, including in particular diffusive approximations and shell-representations of fluid dynamics. In these local dynamics, I will discuss insights from various numerical strategies addressing the underlying non-linear eigenvalue, from shell-time renormalization scheme to bifurcation theory and brute-force optimization.

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## 1. Blow-up problem

- K41 Roughness
- Blow-up problem
- Beale-Kato-Majda
- Leith vs Sabra

## 2. Leith

- Formulation and connection to the singularity problem
- Observations (Anomalies)
- Homoclinic interpretation
- Dombre-Gilson interpretation
- BKM

## 3. Sabra

- Formulation
- (DG) Observations
- Eigenvalue problem
- Sabra hierarchy
- Homoclinic explosions

- Non-universality

## 4. Concluding remarks

- Discrete vs continuous: universal or not Self-similar blowups
- Equivalence between DG and homoclinic formulation
- Hierarchy beyond self-similarity ?

## 1. Blow-up problem

The blow-up problem describes the finite-time development of Holder singularity out of sufficiently smooth initial data. It is believed to be of fundamental importance related to the onset of turbulence in fluid systems. At this stage, let us recall that turbulence is a fluid state characterized in particular by a high degree of roughness. Kolmogorov theory of Navier-Stokes turbulence prescribe infinite gradients and Holder regularity, sustained by a cascade mechanism

$$U_\ell \propto (\epsilon \ell)^{1/3}.$$

In the NS equations, the blow-up scenario is unsettled – see [Gibbon '08].

Criterion exist. For example, the Beale-Kato-Majda criterion states that smooth solutions to the NS can be continued indefinitely unless the integral $\int_0^{t_*} \|\omega\|_\infty$ diverges). This implies that blow-ups must have rates higher than $\|\omega\|_\infty \simeq 1/(t_* - t)$.

Our purpose here is to address the blow-up in simple *cascade models* of turbulence.

The models are

- Leith model: diffusion in Fourier space [Leith '67, Connaughton & Nazarenko '04, Grebenev & al '14, ST & al '15, ST & al '21]
- Sabra model: a system of damped coupled oscillators [Biferale '03, Dombre-Gilson '98, Andersen-al '00, Constantin-al '07, Mailybaev '12]

Leith and Sabra share some essential features:

- Kolmogorov (rough) scaling
- Non-linearity
- Locality

The main structural difference is the discrete vs continuous nature. This will be an issue. We can also point out that shell models usually display intermittency, but not Leith.

Blowup problems map to dynamical system theory. The difference between discrete and continuous will be a difference in the dimensionality. Today, blowups will turn out to be self-similar, meaning that when suitably rescaled, the finite-time blow-up can be seen as a travelling wave in logarithmic space. The blowup speed cannot be determined by dimensional analysis and relates to the presence of anomalous scaling, which we we will define and analyse.

## 2. Leith model

### a. Formulation

*Fourier Space*

$$\partial_t E + \partial_k P = 0, \text{ with } P := -\frac{1}{d+\xi} k^m E^n \partial_k (E/k^{d-1}) \tag{LM-k}$$

$n$ represents the nonlinearity. $d$ is a space-dimensionality. $m$ is an adhoc parameter, which prescribes that the Leith model exhibits Kolmogorov scaling. We set $m = d + (n+1)(\xi+1)$ in terms of the roughness parameter $\xi$. For the sake of this presentation, think about

$$n = 0 \text{ or } n = 1/2, d = 3, \xi = 2/3$$

It is not hard to check that Eq.~(LM-k) has the exact power law solutions

$$E \propto k^{-\xi-1}, P \propto 1 \text{ (K41)} \quad \text{ and } \quad E \propto k^{d-1}, \text{ P=0 (Equilibrium)}$$

*Connection to the singularity problem.*

The physical-space representation of LM uses the variable $U(\ell) = kE(k), \ell = 1/k$. $U$ represents the scaling part of a correlation function (second order increments) and obeys the dynamics (Remember $\partial_k = -\ell^2 \partial_\ell$!)

$$\partial_t U + \ell \partial_\ell \Pi = 0, \text{ with } \Pi(\ell) := -\frac{1}{d+\xi} \ell^{2+n-m} U^n \partial_\ell (\ell^d U) \tag{LM-x}$$

The scaling solutions are

$$U \propto \ell^\xi, \Pi \propto -1 \text{ (K41)} \quad \text{ and } \quad U \propto \ell^{-d}, P = 0 \text{ (Equilibrium)}$$

#### b. Observations

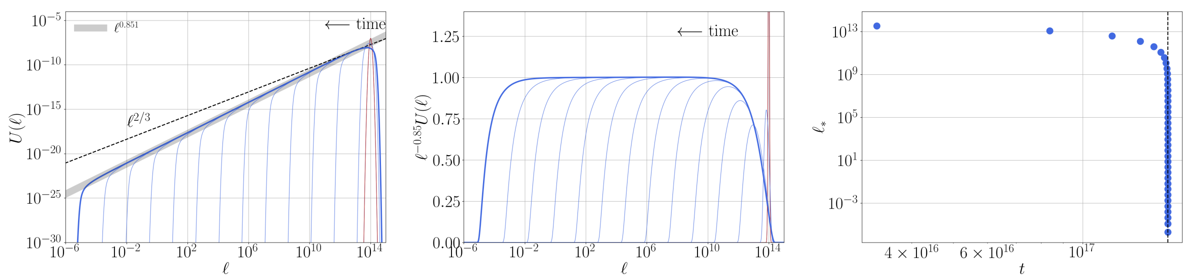


Figure 1: Finite-time blow up featuring anomalous exponent in its trail.

#### c. Homoclinic viewpoint

*Scaling Ansatz.* We look for a non-constant flux solution

$$E(k,t) = k_*^{-x-1} F(\eta), \quad \eta := k/k_*, \quad k_* := (t_* - t)^{-b}$$

$$\text{with } b(x) := \frac{1}{n(\xi - x) + \xi} > 0$$

Under this scaling ansatz, the flux becomes

$$P = k_*^{(n+1)(\xi+1-x)} \Pi(\eta), \ \Pi := -\eta^m \phi^n (\phi/\eta^{d-1})',$$

The profile $F$ is determined by the BV problem

$$-b(xF + \eta F') + \Pi' = 0, \quad \Pi(\eta) = -\eta^m F^n \left(F \eta^{1-d}\right)' \tag{1}$$

with limits

$$F \sim \eta^{-x-1}, \quad \Pi(\eta) \sim \eta^{(n+1)(\xi - x)} \quad \text{at } 0$$

*Homoclinic explosion.* The problem is most conveniently analyzed upon introducing log-similarity variables $F, \Pi, \eta \to f, p, \tau = \dfrac{1}{bn}\log(\eta)$ and the rescaled profiles

$$f(\tau) = \eta^{\alpha} F, \quad p(\tau) = \eta^{\alpha-1}\Pi, \quad \alpha := 1 + \xi\left(1 + \frac{1}{n}\right) = x + 1 + \frac{1}{bn}$$

The BV problem then becomes

$$(nb)^{-1}\dot{f} = \left(\xi\left(1+\frac{1}{n}\right) + d\right) f - p f^{-n}, \quad \text{et} \quad (nb)^{-1}\dot{p} = \xi\left(1+\frac{1}{n}\right) p + b(x)\left((x+d)f - f^{-n}p\right). \tag{2}$$

under the boundary behaviors

- $\tau \to -\infty \quad (\eta \to 0)$:
$$f \propto e^{\tau} \to 0, \quad p \propto e^{(n+1)\tau} \to 0$$

- $\eta \to \infty$:
$$f, p \to 0$$

  The looked-for solution is a homoclinic bifurcation of the autonomous system. In this 2d situation, this observation provides unicity of the self-similar profile. The eigenvalue $x$ can be found from standard techniques.

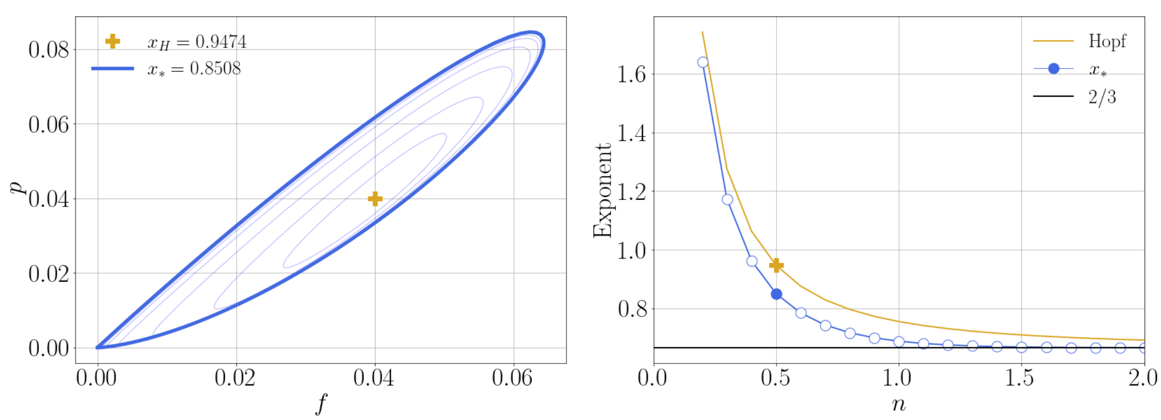


Figure 2: Homoclinic explosion in the Leith model determined by numerical continuations.

**d. Dombre-Gilson (DG) viewpoint**

The autonomous system was obtained by the following sequence

$$\text{Leith} \quad \to \quad \text{ansatz} \quad \to \quad \text{autonomous system} \quad \to \quad \text{Homoclinic explosion}$$

The same final system can be retrieved by following the following and somewhat less formal path

$$\text{Leith} \quad \rightarrow \quad \text{Vorticity} \quad \rightarrow \quad \text{DG rescaling} \quad \rightarrow \quad \text{Traveling wave}$$

Define the vorticity variables $W := k^\alpha E = \ell^{1-\alpha} U$. Then,

1. the Leith Model becomes the homogeneous dynamics

$$\begin{aligned} &\partial_t W + L_1[\Pi] = 0, \quad \Pi = -L_2[W^{n+1}], \\ &\text{for } L_1 = k\partial_k + (1-\alpha)\mathrm{Id}, \quad L_2 := \frac{1}{n+1} k\partial_k + (1-d-\alpha)\mathrm{Id} \end{aligned} \tag{LM-W}$$

2. Under the Dombre-Gilson rescaling (DG),

$$f(\tau,\kappa) := e^{-\tau/n} W, \quad \tau = -\log(t_* - t), \quad \kappa = \log k$$

   the LM-W system becomes

$$\partial_\tau f = \mathcal{L}_1[p] - \frac{1}{n} f = 0, \quad p(\tau,\kappa) := -\mathcal{L}_2[f^{n+1}] \tag{3}$$

   where $\mathcal{L}_{1,2}$ are obtained from $L_{1,2}$ upon replacing $k\partial_k \to \partial_\eta$. Considering traveling wave solutions $f(\tau,\kappa) = \tilde{f}(\kappa - b\tau)$ recovers the autonomous system up to a global rescaling of the similarity variable. **The DG transforms a finite-time blow-up to a traveling wave over an infinite-time window!**

3. It is straightforward to check that the $L_p$ norms of the vorticity blow up for the similarity solutions. Specifically, one obtains for all $p > 0$

$$W_p \propto k_*^{\frac{1}{p}+\frac{1}{bn}}, \qquad W_p := \left( \int_0^\infty W^p dk \right)^{1/p}. \tag{4}$$

This implies the diverging behavior $W_p \propto (t_* - t)^{-b/p-1/n}$. In particular $W_\infty \propto (t_* - t)^{-1/n}$, taking $n = 1/2$ and identifying $W_\infty$ with the squared vorticity, one recovers the BKM rate $W_\infty \propto (t_* - t)^{-1}$. This also provides interpretation of the DG scheme: $f(\tau,\kappa) = e^{-\tau/n} W \propto W/W_\infty$. When normalized by its peak, the front behaves as a wave propagating at speed $b$ across the log-scales – See Fig 3.

4. The similarity profile has infinite energy, coming from ir divergence at $\eta \to 0$. This is consistent with the fact that the anomalous scaling is ... anomalous, in the sense that it cannot be deduced from dimensional analysis and energy conservation!

## 3. Shell model

### a. Setting

1. We will now extend our consideration to the shell model cases. The Sabra model [L'vov-al '98] prescribes the dynamics

$$\dot{u}_n = N_n[u] := ik_n \left( \lambda u_{n+2} u_{n+1}^* - (1+c) u_{n+1} u_{n-1}^* - c\lambda^{-1} u_{n-1} u_{n-2} \right), \quad n \in \mathbb{Z} \tag{Sabra}$$

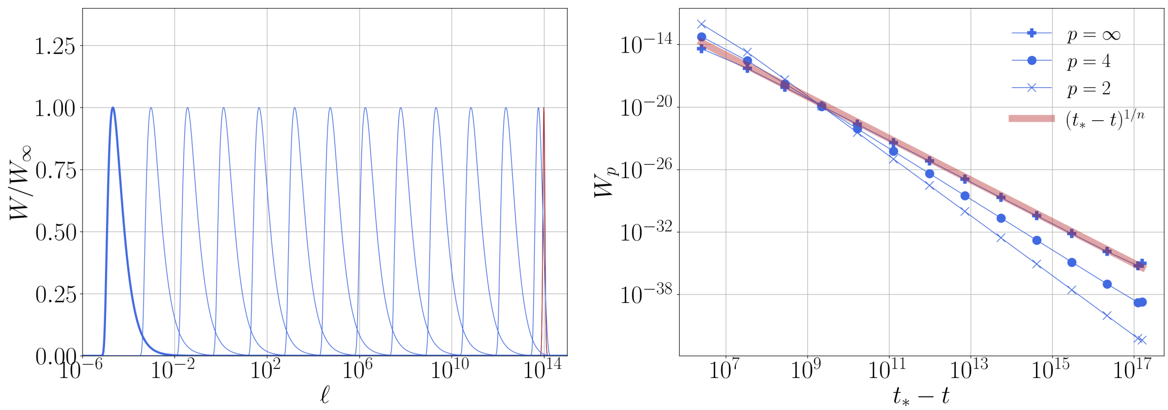


Figure 3: Dombre Gilson interpretation of the Leith blow-up.

in terms of complex velocity variables $u_n$ defined on a geometric progression of scales $\ell_n = k_n^{-1} = \lambda^{-n}$ with $\lambda > 1$ the intershell ratio. The coefficient $c$ is here prescribed to be negative, and we later write it as $c = -\lambda^{-g}$ with $g$ a positive constant. In this form, Sabra preserves $G = \sum_n (-1)^n \lambda^{gn} |u_n|^2$ and $E = \sum_n |u_n|^2$. Think about $g = -1$ for the clasical Sabra dynamics mimicking the Navier-Stokes.

2. From previous numerical works under the DG scheme: The blow-up is self-similar with anomalous exponents $\simeq 0.28$. Specifically, prescribe imaginary dynamics $u_n \in i\mathbb{R}$. The similarity Ansatz reads

$$u_n = -ik_*^{-x}\,\mathcal{U}\left(\eta_n\right), \quad \text{with } \eta_n := k_n/k_*, \ \text{ and } k_* := (t_* - t)^{-\frac{1}{1-x}} \tag{5}$$

in terms of the scaling exponent $x < 1$ and the blowup time $t_* < \infty$. The minus sign is conventional, The autonomous form of the Sabra BVP involves the vorticity

$$\mathcal{W}(\tau_n) = \eta_n \mathcal{U}(\eta_n), \quad \tau_n = \log \eta_n$$

and takes the form of the fixed point problem

$$\dot{\mathcal{W}} = \mathcal{F}\left[\mathcal{W}\right], \quad \mathcal{F}\left[\mathcal{W}\right] = \mathcal{W} + \lambda^{-2}\,\mathcal{W}_{+2\delta}\,\mathcal{W}_{+\delta} - (1+c)\,\mathcal{W}_{+\delta}\,\mathcal{W}_{-\delta} + c\lambda^2 \mathcal{W}_{-\delta}\,\mathcal{W}_{-2\delta} \tag{6}$$

with the shorthand $\mathcal{W}_{j\delta}(\tau) := \mathcal{W}\left(\tau + j(1-x)\log\lambda\right)$, and boundary conditions $\mathcal{W} \underset{-\infty}{\sim} e^\tau \to 0, \quad \mathcal{W} \underset{\infty}{\to} 0$.

3. Promotion to a continous setting: $\mathcal{W} = \mathcal{W}(\tau_n), n \in \mathbb{Z} \to \mathcal{W}(\tau), \tau \in \mathbb{R}$

4. Elementary fixed points : $\mathcal{W}_0 = 0$ and $\mathcal{W}_{\mathrm{H}} = \frac{\lambda^2}{(\lambda^2-1)(1-c\lambda^2)} > 0$.

5. Hopf: $x_{\mathrm{Hopf}} = 1 - \frac{(\lambda^2-1)(1-c\lambda^2)}{(1-c\lambda^4)\log\lambda} \frac{\arccos\rho}{(1+2\rho)\sqrt{1-\rho^2}} < 1$, for $\rho$ the root with norm $<1$ of

$$\rho^2 + \left(\frac{1}{2} - R\right)\rho + \frac{R}{2} - 1 = 0, \quad R := \frac{c+1}{c\lambda^2 + \lambda^{-2}}. \tag{7}$$

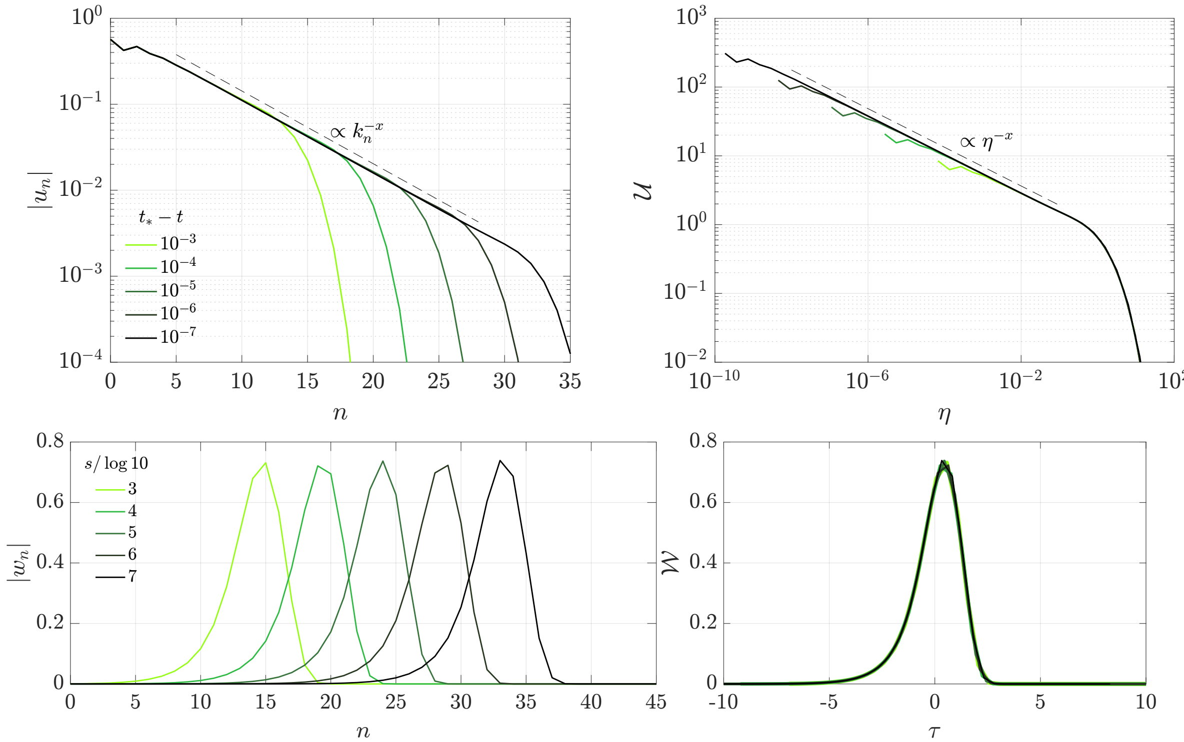


Figure 4: Numerical observations of Sabra dynamics.

## b. Sabra Hierarchy

The presence of positive and negative delays make the fixed point equation cumbersome to solve! To retrieve a dynamical system framework, we Taylor-expand in the bookkeeping parameter $\delta = (1-x)\log\lambda$ as

$$\mathcal{W}_k(\tau) = \sum_{i=0}^{M} \frac{k^i \delta^i}{i!} \mathcal{W}^{(i)}(\tau) + O(\delta^{M+1}), \tag{8}$$

which in turn yields for the $\mathcal{F}$ -functional

$$\mathcal{F}[\mathcal{W}] = \mathcal{W}^{(0)}(\tau) + \frac{1}{2}\sum_{k=0}^{M} \delta^k \sum_{i+j=k} \sigma_{ij} \mathcal{W}^{(i)}(\tau)\mathcal{W}^{(j)}(\tau) + O(\delta^{M+1}) \tag{9}$$

In terms of $X_k = \delta^k \mathcal{W}^{(k)}$, and considering successive truncations at order $O(\delta^M)$, we obtain the Sabra hierarchy, %

$$\delta\, \dot{X}_k(\tau) = X_{k+1}, \quad 0 \le k \le M-2, \quad \sigma_{0M}\delta X_0 \dot{X}_{M-1}(\tau) = G_M, \tag{10}$$

with boundary conditions $\mathbf{X}(\tau) \underset{\pm\infty}{\to} 0$. With the singular time $\theta = \int_0^\tau \frac{d\tau'}{\delta X_0(\tau')}$, we obtain the (non-singular) hierarchy

$$\begin{cases} M=2: \quad X_0' = X_0 X_1,\ \sigma_{02} X_1' = -X_0 + \dfrac{X_1}{\delta} - \dfrac{1}{2}\sigma_{00} X_0^2 - \sigma_{01} X_0 X_1 - \dfrac{1}{2}\sigma_{11} X_1^2 = G_2, \\ M=3: \quad X_0' = X_0 X_1,\ X_1' = X_0 X_2,\ \sigma_{03} X_2' = G_2 - \sigma_{02} X_0 X_2 - \sigma_{12} X_1 X_2 = G_3, \quad etc. \end{cases} \tag{11}$$

$$\text{with } G_{M+1} = G_M - \sigma_{0M} X_0 X_M - \frac{1}{2} \sum_{\substack{i+j=M+1 \\ 1\le i,j\le M}} \sigma_{ij} X_i X_j, \tag{12}$$

and the coefficients

$$\sigma_{ij} = \frac{\lambda^{-2}(2^i + 2^j)}{i!j!} \left( (-1)^{i+j} c\lambda^4 - \frac{(-1)^i + (-1)^j}{2^i + 2^j} (1+c)\lambda^2 \right)$$

**c. Bifurcation scheme**

- Local analysis:
  - Set of fixed points independent from $M$: the origin and $\mathbf{X}_\mathrm{H} = (\mathcal{W}_\mathrm{H}, 0, 0, \dots)$ .
  - $\mathbf{X}_\mathrm{H}$ is isolated while the origin is part of a wider set

$$\mathcal{Z}_0^{(M)} = \left\{ \mathbf{X} = (0, \mathbf{X}_+), \text{ with } \mathbf{X}_+ \in \mathbb{R}^{M-1} \;\&\; \mathcal{G}_M\left[(0, \mathbf{X}_+)\right] = 0 \right\} \tag{13}$$

- At all orders, we evidence the scenario

  stable fixed point $\to$ Hopf bifurcation $\to$ stable limit cycle $\to$ homoclinic orbit.

- Note however the degeneracy of the homoclinic cycle!

## 4. Beyond fundamentals

Direct optimization of the fixed point equation with stochastic gradient descent suggest non-unicity of the blowup, with discrete $x_* > 0.28$

Those self-similar profiles are seen in DNS.

## 5. Concluding remarks

- Analogy Leith/Sabra: Blowup characterized with DG-traveling wave/homoclinic explosion
- Sabra : richer dynamics and non-universality of blowup (Non uniqueness).
- The sign of the anomaly is model-dependent.
- Three methods: DG, ML, hierarchy.
- The perturbative strategy proves highly efficient, with exponentially fast convergence in $M$ towards the fundamental solution. The ML substantiates underlying complexity by providing a simple tool to detect nontrivial and unstable solutions. Both strategies might be used in more complex framework, beyond the 3D Sabra model.
- Beyond self-similar blowups, the Dombre-Gilson scheme and its variants has revealed that shell models could exhibit a variety of blowup scenarios, including the possibility of chaotic blowups describing the asymptotic finite-time convergence onto a self-similar chaotic attractor. Can transition to chaos be analysed with a hierarchical strategy?

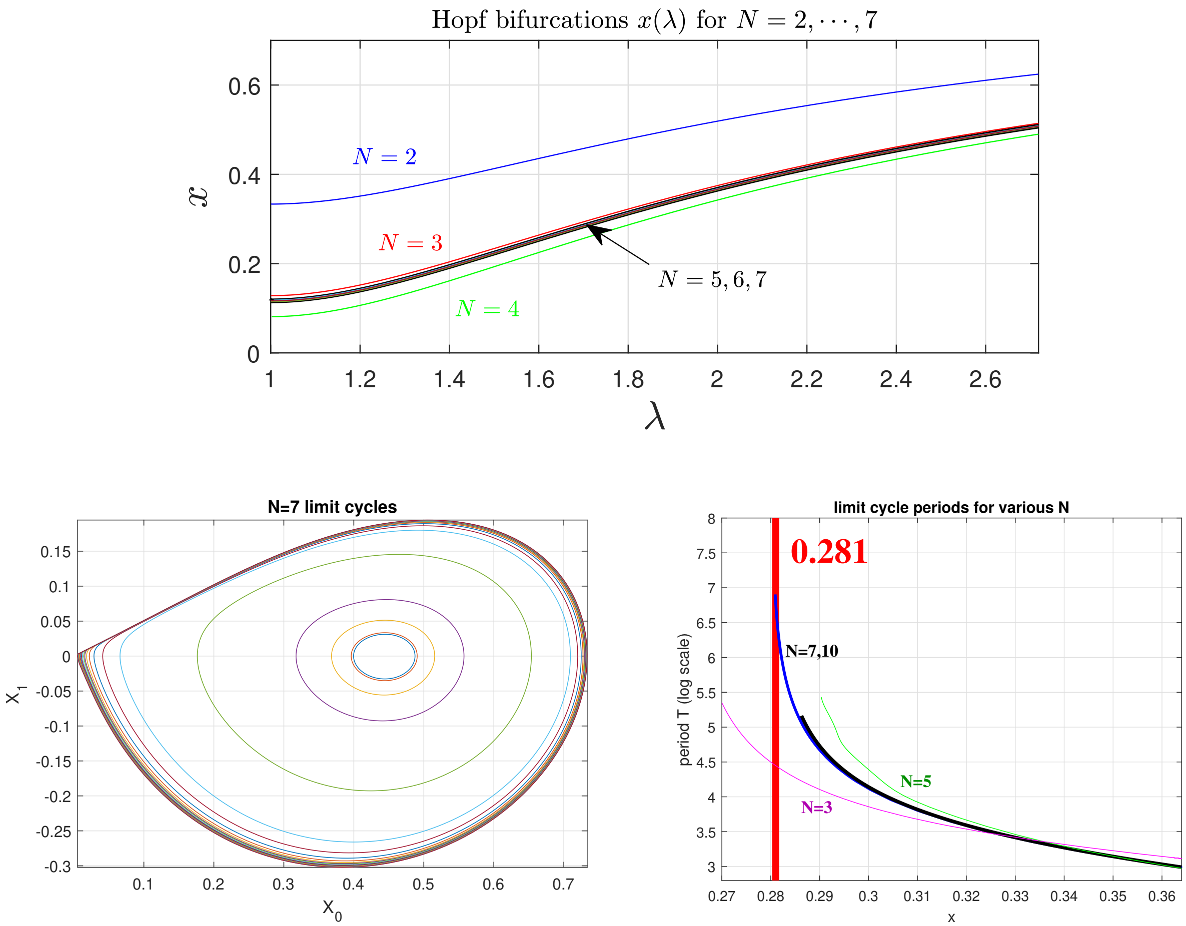


Figure 5: Convergence of the homoclinic explosion.

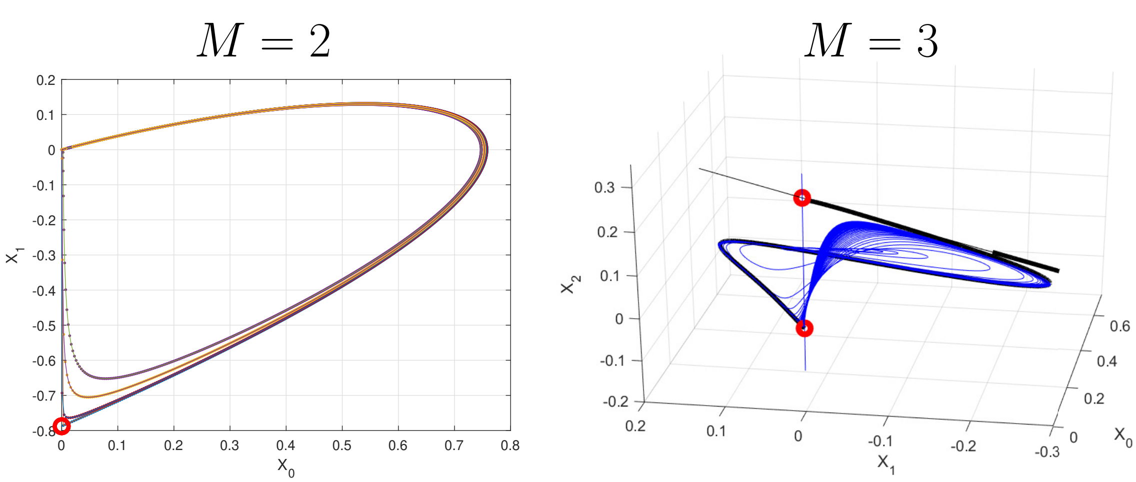


Figure 6: Homoclinic degeneracy.

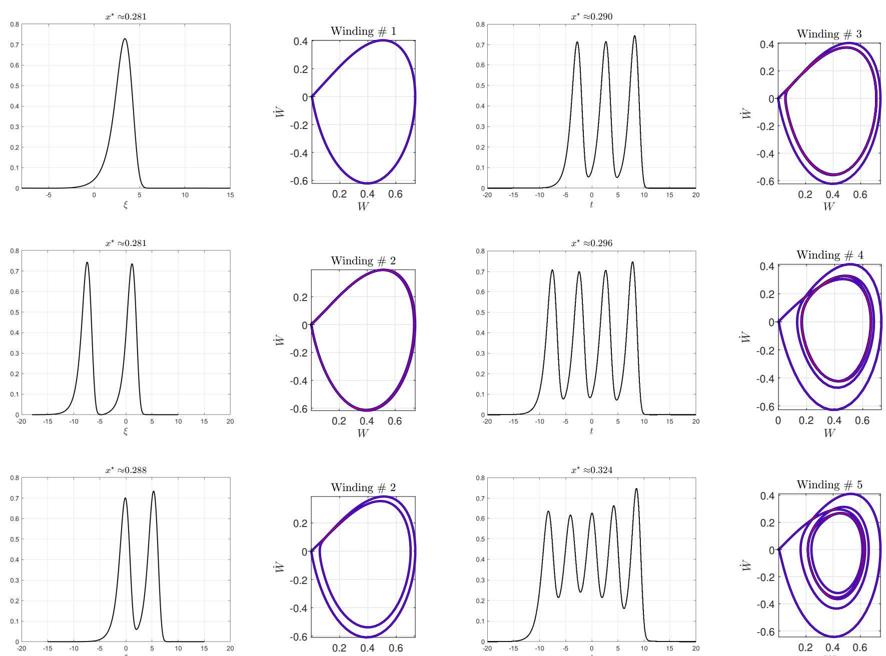

Figure 7: (Non-unique) solitonic profiles.