From ZERO-MODE INTERMITTENCY to HIDDEN SYMMETRY in RANDOM SCALAR ADVECTION

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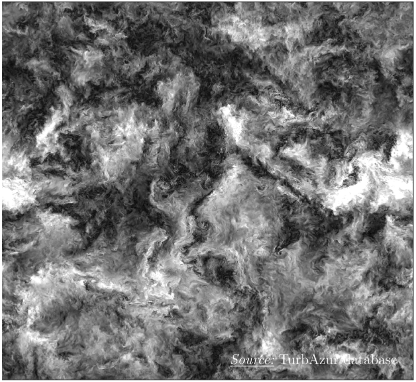
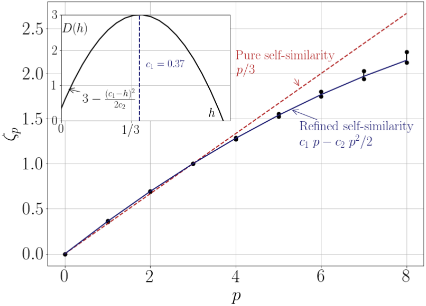
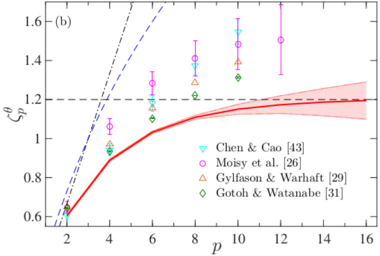


$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \kappa \Delta \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \mathbf{f}_\theta + \kappa \Delta \theta$$

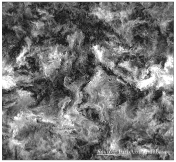




From Iyer & al, PRL ('18)

$$\left\langle |\delta u_{\|}|^p \right\rangle \propto \ell^{\zeta_p}$$

$$\left\langle |\delta \theta|^p \right\rangle \propto \ell^{\zeta_p^\theta}$$

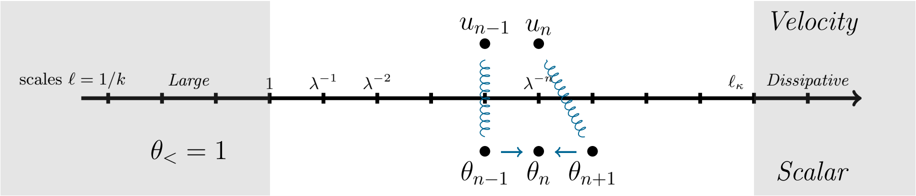
	Navier-Stokes	Passive Scalar
		
Kinematic origin		
Refined self-similarity	Kolmogorov ('61) ...	Stolovitzky & al ('95), Warhaft ('00)
Multifractal framework	Parisi-Frisch ('85) ...	Prasad & al ('88), Ruiz-Chavarria & al ('96), Gotoh & Watanabe ('15), Schmitt & Huang ('16) , Iyer & al ('18) ...
Dynamical origin		
Zero-mode theory Statistical conservation laws	?	Kraichnan flows O('90 -'00)
Hidden symmetry	AAM ($\geq$ '21) AAM & ST ('22)	?

Layout

1. **Kraichnan-Wirth-Biferale (KWB) dynamics**
2. Zero Modes
3. Hidden symmetry
4. Inertial Hidden symmetry
5. Concluding remarks

Kraichnan-Wirth-Biferale (KWB) model

Jensen & al ('92), Biferale & Wirth ('96 '07), Andersen & Muratore-Ginanneschi ('99)

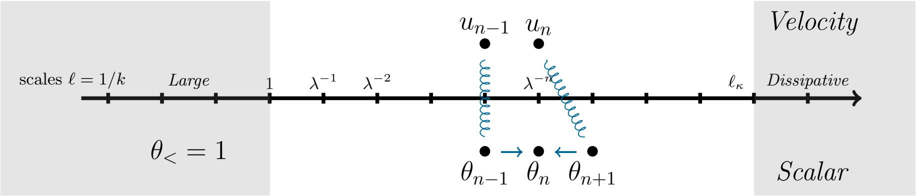


Random advection

$$\left(\frac{d}{dt} + \kappa k_n^2\right)\theta_n = k_n\theta_{n+1}u_n - k_{n-1}\theta_{n-1}u_{n-1},$$

u : Random (Gaussian) flow

Jensen & al ('92), Biferale & Wirth ('96 '07), Andersen & Muratore-Ginanneschi ('99)



Kraichnan limit: $\langle u_n(t) u_m(t+\tau)\rangle \propto \ell_n^{\xi}\, \delta_{nm}\, \delta(\tau)$

- Ito dynamics: $d\theta_n = \mathcal{N}_n[\theta, dw] + (I_n + B_n - D_n)\theta_n\, dt$
- Advection: $\mathcal{N}_n[\theta, dw] = \gamma^n \theta_{n+1} dw_n - \gamma^{n-1}\theta_{n-1} dw_{n-1}, \quad \gamma := \lambda^{1-\xi/2}.$
- Drift: $I_n = -\frac{\gamma^{2n}+\gamma^{2n-2}}{2}, \quad B_n = \frac{\delta_{n1}}{2}, \quad D_n = \kappa\lambda^{2n}$

Scalar Balance

$$\frac{d\left\langle \theta_n^2 \right\rangle}{dt} + \Pi_n - \Pi_{n-1} = -2D_n \left\langle \theta_n^2 \right\rangle,$$

with the scalar flux from shell n to $n+1$ due to advection:

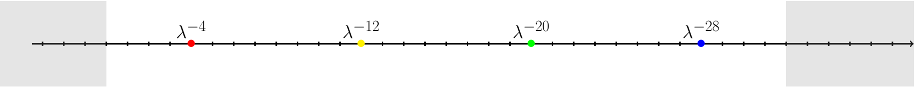
$$\Pi_n = \begin{cases} 1, & n = 0; \\ \gamma^{2n}\left\langle \theta_n^2 \right\rangle - \gamma^{2n}\left\langle \theta_{n+1}^2 \right\rangle, & n \geq 1 \end{cases}$$

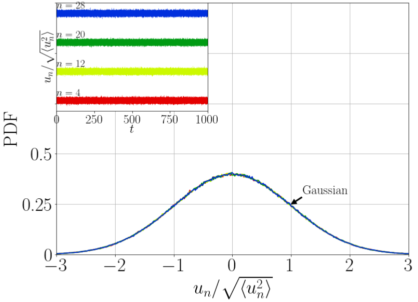
Inertial scales

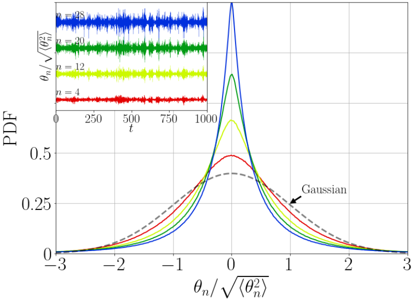
- Standard steady-state phenomenology with constant flux cascade

$$\left\langle \theta_n^2 \right\rangle \propto \ell_n^{2-\xi}, \quad 0 \ll n \ll n_\kappa, \quad n_\kappa := \frac{1}{\xi}\log_\lambda \frac{1}{\kappa}.$$

- Math: $t \to \infty, \quad \kappa \to 0, \quad n \to \infty$ from left to right.

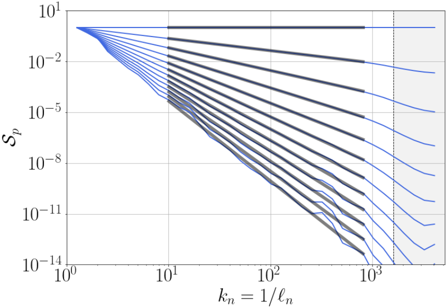


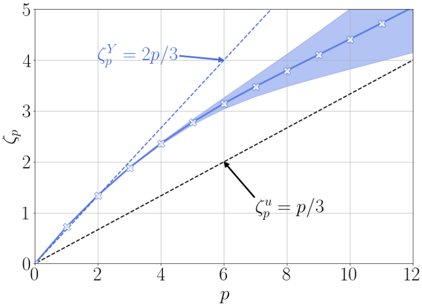




Gaussian velocity $\rightsquigarrow$ Non-Gaussian scalar!

$$\mathcal{S}_p(\ell_n) := \lim_{T\to\infty} T^{-1}\int_0^T \langle|\theta_n|^p\rangle\, dt \quad \propto \ell_n^{\zeta_p}$$





The KWB dynamics provides a minimal random model for scalar intermittency!

Layout

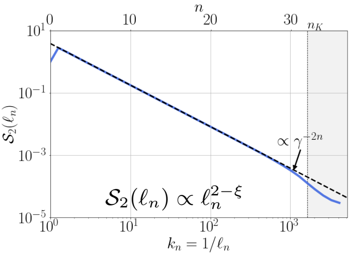
1. Kraichnan-Wirth-Biferale (KWB) dynamics
2. **Zero Modes**
3. Hidden symmetry
4. Inertial Hidden symmetry
5. Concluding remarks

$$\mathcal{S}_2 := \lim_{T\to\infty} T^{-1} \int_0^T \langle |\theta_n|^2 \rangle \, dt$$

- Inertial-range recursion

$$0 = \mathcal{S}_2(\ell_{n-1}) - (1+\gamma^2)\mathcal{S}_2(\ell_n) + \gamma^2 S_2(\ell_{n+1}),$$

$$\text{with boundary conditions} \begin{cases} \mathcal{S}_2(\ell_n) \underset{\infty}{\to} 0 \text{ (small-scale)} \\ \mathcal{S}_2(\ell_0) = 1 \text{(large scale)}. \end{cases}$$



- Zero-mode interpretation

$$\underbrace{\begin{pmatrix} \ddots & \ddots & & \\ \ddots & -(1+\gamma^2) & \gamma^2 & \\ & 1 & -(1+\gamma^2) & \ddots \\ & & \ddots & \ddots \end{pmatrix}}_{M_2} \begin{pmatrix} \vdots \\ \mathcal{S}_2(\ell_n) \\ \mathcal{S}_2(\ell_{n+1}) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

$\Rightarrow$ Kolmogorov scaling is a (trivial) zero-mode.

$$\mathcal{S}_4(\ell_n) := \sigma_{n0}, \qquad \sigma_{nl} = \lim_{T\to\infty} T^{-1} \int_0^T \left\langle \theta_n^2 \theta_{n+l}^2 \right\rangle dt.$$

- Zero-mode interpretation: $\mathcal{M}_4 \sigma = 0$

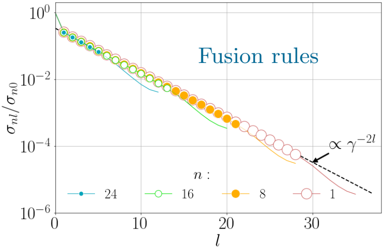
- Inertial-range recursion

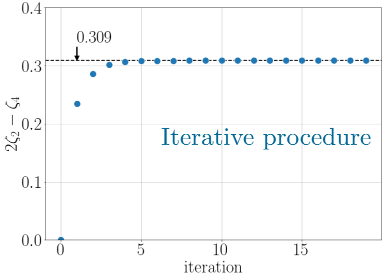
$$\begin{aligned} 0 = \quad & b_{-l}\sigma_{(n+1)(l-1)} \\ & + b_l \gamma^{-2} \sigma_{n(l-1)} - a_l \sigma_{nl} + b_l \sigma_{n(l+1)} \\ & \qquad + b_{-l} \gamma^{-2} \sigma_{(n-1)(l+1)} \end{aligned} \quad \text{for} \quad \begin{cases} a_l = \gamma^l + \gamma^{-l-2} + \gamma^{-l} + \gamma^{l-2} + 4\gamma^{-l}\delta_{l1}, \\ b_l = \gamma^l + 2\delta_{l0}. \end{cases}$$

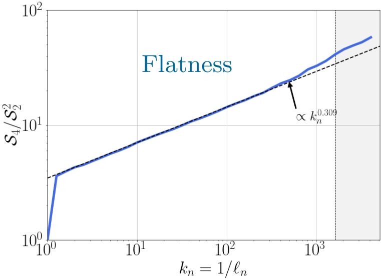
- Benzi Ansatz Benzi & al ('97):
  - $S_4(\ell_n) = \sigma_{n0} \underset{\infty}{\propto} \ell_n^{\zeta_4}$ (Scaling Ansatz)
  - $\sigma_{nl} = C_l \sigma_{n0}$ with $C_l \underset{\infty}{\to} 0$ & $C_0 = 1$ (Fusion Rule)

The Benzi Ansatz yields the fixed point:

$$\zeta_4 = \log_\lambda \left(\frac{1+\gamma^2}{3C_1(\zeta_4)} - \gamma^2 \right).$$







Statistical conservation laws

The ideal KWB dynamics (statistically) preserve

$$\Gamma_2 := \sum_{n\in\mathbb{Z}} \gamma^{-2n} \left\langle \theta_n^2 \right\rangle, \qquad \Gamma_4 = \sum_{n\in\mathbb{Z}} \sum_{l\geq 0} c_l r^n \left\langle \theta_n^2 \theta_{n+l}^2 \right\rangle,$$

where $c_l = 6C_l - 5\delta_{l0}$ and $r = \lambda^{-\zeta_4}$.

Proof: The vectors $(\gamma^{-2n})_n$ and $(r^n c_l)_{n,l}$ are, respectively, zero-modes of the dual operator $\mathcal{M}_2^\dagger$ and $\mathcal{M}_4^\dagger$.

Observations

- The ideal statistical conservation of Γ_2 is dual to the Kolmogorov scaling $\mathcal{S}_2(\ell_n) = \gamma^{-2n}$.
- Relevance of the Benzi Ansatz here reflects a degeneracy of the dynamics.

Limitations of the zero-mode theory

- Ansatz on scaling and fusion rules.
  L'Vov & Procaccia ('96), Fairhall & al ('97), Biferale & al ('99), Friedrich & al ('18)

- Duality between zero modes and statistical conservation laws.
  Beyond linear setting: Angheluta & al ('06), Arad & al ('01)

- Even-order structure functions.
  Beyond p=4 and white-in-time setting: Andersen & Muratore-Ginanneschi ('99)

1. Kraichnan-Wirth-Biferale (KWB) dynamics
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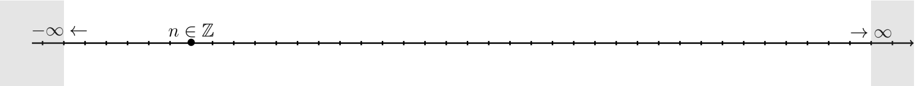
Ideal KWB: $d\theta_n = \mathcal{N}_n[\theta, dw] + I_n \theta_n \, dt$

$$\begin{cases} \mathcal{N}_n[\theta, dw] = \gamma^n \theta_{n+1} dw_n - \gamma^{n-1} \theta_{n-1} dw_{n-1}, \\ I_n = -\dfrac{\gamma^{2n} + \gamma^{2n-2}}{2}, \quad \gamma := \lambda^{1-\xi/2} \end{cases}$$

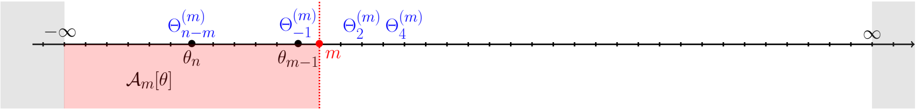


Scale invariance

The ideal KWB is statistically invariant under the rescaling

$$n \mapsto n \pm 1, \quad \theta_n \mapsto \lambda^h \theta_{n\pm 1}, \quad w_n \mapsto \gamma^{\pm 1} w_{n\pm 1}, \quad t \mapsto \gamma^{\pm 2} t, \quad h \in \mathbb{R}$$

In other words, $\theta'_n(t') := \lambda^h \theta_{n-1}(t/\gamma^2)$ *solves the ideal KWB for the Wiener noise* $w'_n(t') = \gamma w_{n-1}(t/\gamma^2)$

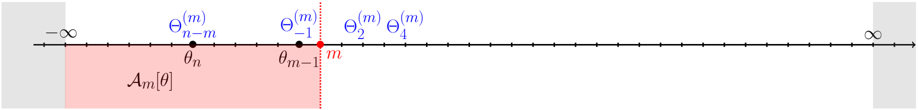


Shell-time rescaling: $t, w, \theta \mapsto \tau^{(m)}, W^{(m)}, \Theta^{(m)}$

$$\Theta^{(m)}_N = \frac{\theta_{m+N}}{\mathcal{A}_m[\theta]}, \quad W^{(m)}_N = \gamma^m w_{m+N}, \quad \tau^{(m)} = \gamma^{2m} t.$$

with

- $m > 0$ (reference shell)
- $\mathcal{A}_m[\theta] = \sqrt{\theta_m^2 + \alpha\theta_{m-1}^2 + \alpha^2\theta_{m-2}^2 + \cdots}$ (Scalar amplitude)
- $0 < \alpha \ll 1$



Hidden KWB

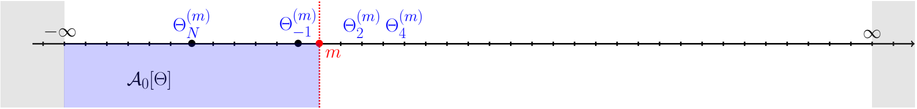
Under the hidden rescaling, the ideal KWB rescales into

$$d\Theta^{(m)}_N = \circ\Lambda_N\left[\Theta^{(m)}, \mathcal{N}\left[\Theta^{(m)}, dW^{(m)}\right]\right], \quad N \in \mathbb{Z},$$

with

$$\Lambda_N[\Theta, V] = V_N - \Theta_N \sum_{J\leq 0} \alpha^{-J}\Theta_J V_J.$$

Obs: The Hidden KWB is nonlinear!



Change of reference shell

The Hidden KWB is statistically invariant under

$$m \mapsto m \pm 1: \qquad \Theta \mapsto a^{\pm 1}[\Theta], \quad W \mapsto b^{\pm 1}[W], \quad \tau \mapsto \gamma^{\pm 2}\tau.$$

with

$$a_N^{+1}[\Theta] = \frac{\Theta_{N+1}}{\sqrt{\alpha + \Theta_1^2}}, \quad a_N^{-1}[\Theta] = \sqrt{\frac{\alpha}{1-\Theta_0^2}}\,\Theta_{N-1}, \quad b_N^{\pm 1}[W] = \gamma^{\pm 1} W_{N\pm 1}.$$

Obs: The hidden scale invariance does not depend on h!

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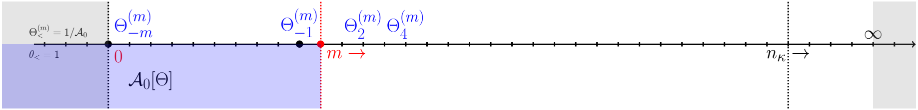


In the presence of forcing & dissipation, the hidden scale invariance is broken. The limit $t \to \infty, n_\kappa \to \infty$ defines

(finite-m) stationnary measures

$$\mathbb{P}^{(m)}(d\Theta) = \lim_{T\to\infty} T^{-1} \int_0^T d\tau^{(m)} \left\langle \mathbb{1}_{\Theta^{(m)}(\tau^{(m)}) \in d\Theta} \right\rangle$$

$$\mathbb{P}^{(m\pm1)} = a_\sharp^{\pm1} \mathbb{P}^{(m)}$$

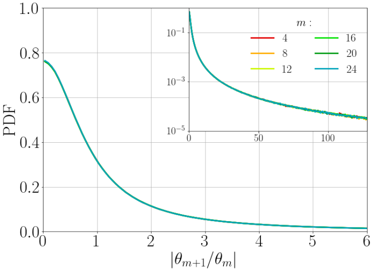
Hypothesis: Inertial hidden scale invariance (inertial HS)

$$\mathbb{P}^{(m)} \underset{\infty}{\rightarrow} \mathbb{P}^{\infty}, \qquad \mathbb{P}^{\infty} = a_\sharp^{\pm1} \mathbb{P}^{\infty}$$

Push-forwards of $\mathbb{P}^{(m)}$ are inertial HS .

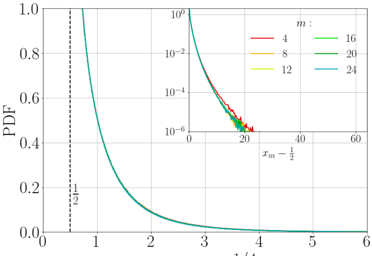
Kolmogorov Multipliers

- $\mathbb{P}^{(m)}_{\mathcal{W}} = \mathcal{W}_{0\sharp}\mathbb{P}^{(m)}, \quad \mathcal{W}_0[\Theta] := \left|\dfrac{\Theta_0}{\Theta_{-1}}\right|$
- $w_m(t) = \left|\dfrac{\theta_m}{\theta_{m-1}}\right| = \mathcal{W}_0[\Theta^{(m)}(\tau^{(m)})]$

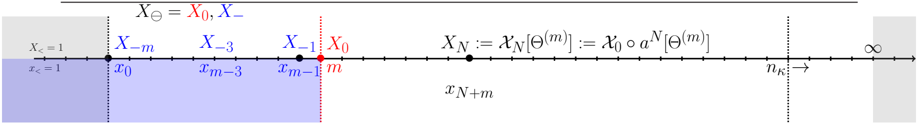


Generalized Multipliers

- $\mathbb{P}^{(m)}_{\mathcal{X}} = \mathcal{X}_{0\sharp}\mathbb{P}^{(m)}, \quad \mathcal{X}_0[\Theta] := \sqrt{\dfrac{\alpha}{1-\Theta_0^2}}$
- $x_m(t) = \left|\dfrac{A_m}{A_{m-1}}\right| = \mathcal{X}_0[\Theta^{(m)}(\tau^{(m)})]$



Consequence 2: Scaling exponents



$$\Sigma_p(\ell_n) := \lim_{T\to\infty} T^{-1} \int_0^T \langle \mathcal{A}_m^p[\theta] \rangle \, dt \quad \text{for} \quad p \in \mathbb{R}.$$

- Multiplicative form

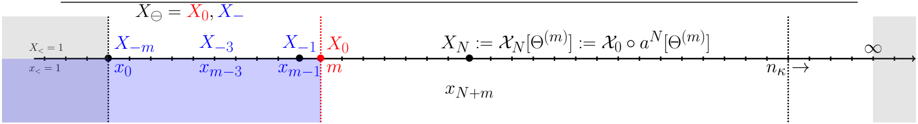
$$\mathcal{A}_m[\theta(t)] = A_0 \prod_{n=1}^{m} x_n(t),$$

$$\Sigma_p(\ell_m) = \int \mu_p^{(m)}(dX_\ominus), \quad \mu_p^{(m)}(dX_\ominus) = A_0^p \left(\prod_{N=1-m}^{0} X_N^p \right) \mathbb{P}_{\mathcal{X}}^{(m)}(dX_\ominus). \quad (1)$$

- Recursion

$$\mu_p^{(m)} = \mathcal{L}_p^{(m)}[\mu_p^{(m-1)}], \quad \mathcal{L}_p^{(m)} : \mu(dX_\ominus) \mapsto X_0^p \, \mathbb{P}_{\mathcal{X}}^{(m)}(dX_0|X_-) \, \mu(dX_-), \quad (2)$$

with boundary condition $\mu_p^{(0)}(dX_\ominus) \sim$ Dirac measure



$$\Sigma_p(\ell_n) := \lim_{T\to\infty} T^{-1} \int_0^T \langle \mathcal{A}_m^p[\theta] \rangle \, dt \quad \text{for} \quad p \in \mathbb{R}.$$

- Inertial Hidden symmetry

$$\mathcal{L}_p^{(m)} \to \mathcal{L}_p^\infty, \quad \mathcal{L}_p^\infty : \mu(dX_\ominus) \mapsto X_0^p \, \mathbb{P}_{\mathcal{X}}^\infty(dX_0|X_-) \, \mu(dX_-),$$

- Perron-Frobenius mode

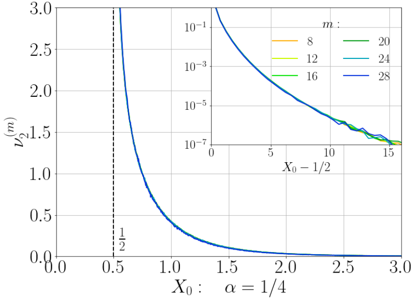
$$\mathcal{L}_p^\infty[\nu_p] = \lambda_p \, \nu_p, \quad \int \nu_p = 1, \quad \lambda_p > 0 : \text{Spectral radius}$$

- Inertial scaling exponents: $\mu_p^{(m)} \sim c_p \lambda_p^m \, \nu_p,$

$$\Sigma_p(\ell_m) \sim c_p \lambda_p^m = c_p \left(\frac{\ell_m}{\ell_0}\right)^{\xi_p}, \quad \xi_p = -\log_\lambda \lambda_p,$$

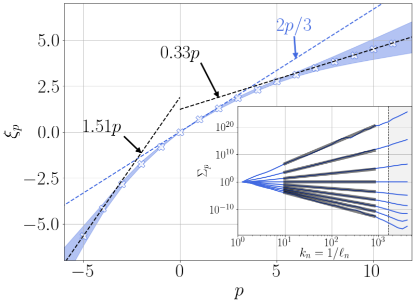
Consequence 2: Scaling exponents (Numerical observations)

Perron-Frobenius



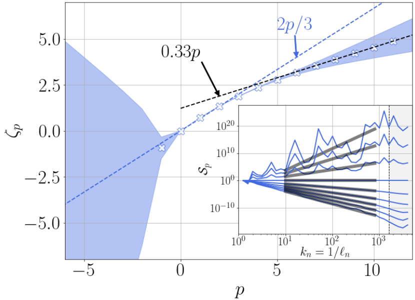
$$\nu_2^{(m)}(dX_0) = \frac{1}{\Sigma_2(\ell_m)} \int_{X_-} \mu_2^{(m)}(dX_0, dX_-)$$

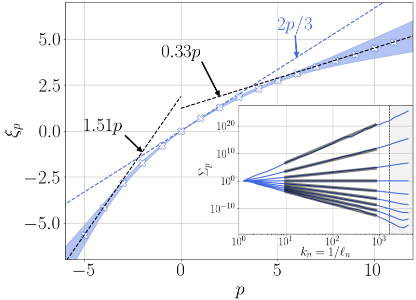
Exponents



$$\Sigma_p(\ell_n) \propto \ell_n^{\xi_p}$$

ζ_p vs ξ_p





$\mathcal{S}_p(\ell_n) \propto \ell_n^{\zeta_p}$

$\Sigma_p(\ell_n) \propto \ell_n^{\xi_p}$

Consequence 3: Fusion Rules

$$\sigma_{ml} := \lim_{T\to\infty} T^{-1} \int_0^T \left\langle \theta_m^2 \theta_{m+l}^2 \right\rangle dt$$

- Bridging relation: $|\theta_m| = \sqrt{1 - \frac{\alpha}{x_m^2}}\, A_0 \prod_{n=1}^m x_n.$

$$l \geq 0 : \quad \sigma_{m(-l)} = \int \left(1 - \frac{\alpha}{X_0^2}\right)\left(1 - \frac{\alpha}{X_{-l}^2}\right)\left(\prod_{j=0}^{l-1} X_{-j}^{-2}\right) \mu_4^{(m)}(dX_\ominus).$$

- Perron-Frobenius: $\mu_4^{(m)} \sim c_4 \lambda_4^m \nu_4 \qquad \sigma_{m(-l)} = \hat{c}_{-l} \left(\frac{\ell_m}{\ell_0}\right)^{\xi_4}$

Fusion rules

$$\sigma_{ml} = C_l \sigma_{m0}, \quad C_l = \begin{cases} \hat{c}_l / \hat{c}_0, & (l \leq 0) \\ \lambda^{-l\xi_4} \hat{c}_{-l} / \hat{c}_0. & (l \geq 0) \end{cases}$$

Layout

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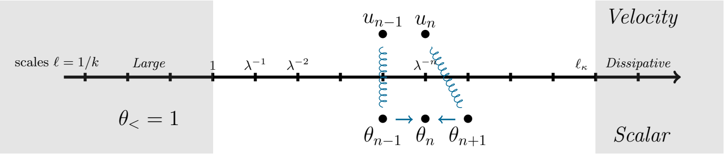
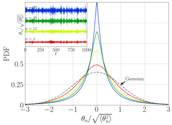
4. Inertial Hidden symmetry

5. **Concluding remarks**

From zero-mode intermittency to hidden symmetry





Zero-Mode	Hidden Symmetry
$\mathcal{M}_{2n}\ \langle\theta\theta\cdots\theta\rangle = 0$	$d\Theta_N^{(m)} = \circ\Lambda_N\left[\Theta^{(m)}, \mathcal{N}\left[\Theta^{(m)}, dW^{(m)}\right]\right],$
Even-order exponents Conservation laws Partly computational	Multipliers Inertial exponents Fusion rules Not specific to linear or random setting
Non-linear settings? Non-perturbative/Ansatz-free computations?	Existence of limit measure ? Computation?

ArXiv: `https://arxiv.org/abs/2402.04198`