# Turbulence of generalized flows in two-dimensions


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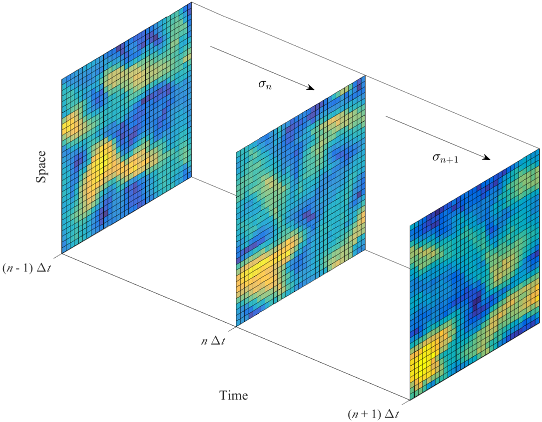
*Arxiv preprint: TBA*




**Workshop on Mathematical and computational problems of Incompressible Fluid dynamics**

**August 10-11, 2018**

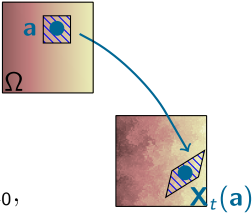
# Layout

Lagrangian variational principles

Numerical construction of generalised flows

2D examples

# Lagrangian variational formulation of ideal fluid motions




ARNOLD, 1966

- **Lagrangian map** : $\mathbf{a} \mapsto \mathbf{X}_t(\mathbf{a}) \in \mathit{Sdiff}(\Omega)$

- **Variational Principle** :

$$\mathcal{A}_{t_0,t_\mathrm{f}} := \int_{t_0}^{t_\mathrm{f}} \mathcal{E}[\mathbf{X}_t]\,\mathrm{d}t \longrightarrow \inf \quad \text{prescribing} \quad \begin{cases} \mathbf{X}_0, \\ \mathbf{X}_f, \\ \det \mathbf{X}_t \equiv 1. \end{cases}$$

- **Euler equations** through geodesics : $\ddot{\mathbf{X}}_t = -\nabla p$

- **Ideal invariants** through Noether Charge:

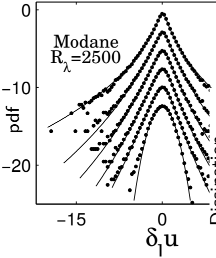
$$\delta Q = \left[ \int_{\mathcal{D}} \mathbf{da}\; \boldsymbol{\pi}(\mathbf{a}) \cdot \delta \mathbf{X}_t(\mathbf{a}) - \mathcal{H} \delta t \right]_{t_0}^{t_f}$$

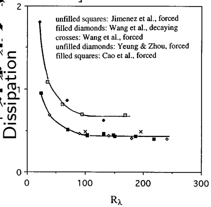
$\Rightarrow$ Widespread applications for geo-physical/plasma modeling : SALMON,1983, MORRISON,1998,...

**But** (i) Solutions may not exist EBIN & MARSDEN,1970, SHNIRELMAN, 1987
(ii) Restricted to classical solutions of the Euler equation

# Turbulence modeling and Euler equations

## Physical evidence of a "turbulent measure" :




From CHEVILLARD ET AL, 2012

SREENIVASAN, 1998

- **Navier Stokes :** $\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = f + \nu \nabla^2 \mathbf{v}$
- **Turbulent limit :** Joint limit $t \to \infty$ and $\nu \to 0$

## Which distributional Euler solutions to describe high-Reynolds motions ?

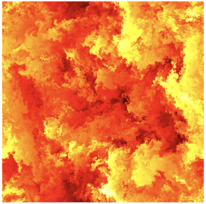
- **Scratch construction** of "turbulent mimicking" solutions to Euler, but not obtained as a limit $\nu \to 0$, and in general non-unique.
  $\Rightarrow$ Examples are the dissipative solutions of SCHEFFER, 1993; DE LELLIS & SZÉKELYHIDI, 2012, ISETT, 2016; BUCKMASTER & AL , 2017 in connection to Onsager's conjecture.
- **Candidate limits** $\nu \to 0$ (*e.g*, DiPerna-Majda measure-valued solutions)

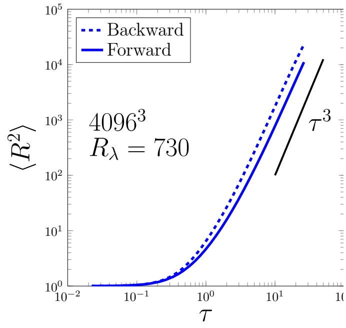
# A Lagrangian turbulent hallmark : intrinsic stochasticity of trajectories

**Roughness** of the velocity field $\Longrightarrow$ **Intrinsic impredictibility** of the transport

$$\delta v_{\parallel}(r) \sim r^h, \qquad h < 1$$

*Kolmogorov 41 :*
$h = \frac{1}{3}$






Turbulent transport is **spontaneously stochastic**

**Fixed realization** of velocity $\Longrightarrow$ Lagrangian **transition probabilities**

Gawedzki,2001, Le Jan & Raimond, 2004 ...

# Generalized variational principle Brenier, 1989, 1999

- **Generalized flow** : probability measure on the **Lagrangian paths**

$$t \to \mathbf{Z}(t) \in \Omega^{[t_0, t_f]}$$

- **Generalized variational principle** :

$$\mathcal{B}[\gamma] := \int \gamma\, [\mathcal{D}\mathbf{Z}]\, \mathcal{E}[\mathbf{Z}] \longrightarrow \inf \quad \text{prescribing} \begin{cases} \gamma(dZ_0, dZ_f) \\ \gamma(dZ_t) = Lebesgue \end{cases}$$

- **Desirable features**:
  1. Existence of optimizers guaranteed by doubly-stochastic boundary coupling.
  2. For determistic coupling given by a classical solution to Euler, classical solutions to Euler are retrieved for small enough $t_f - t_0$.
  3. Non-deterministic solutions exist, with formal link to DiPerna–Majda distributional solutions for small $t_f - t_0$
  4. Dissipative Euler solutions can be constructed (“sticky flows” Shnirelman,1999)

# Questions from the turbulence modeling perspective

1. Are generalized flows physically relevant ?

2. Do they exhibit turbulent features ?

3. Can generalized variational formulations be of relevance to describe inertial-range/coarse-grained dynamics ?

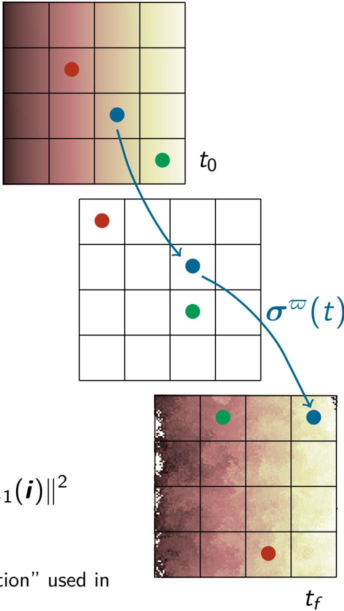
# Layout

# Numerical construction of generalised flows

## Coarse-graining via random permutations

- **Space-time discretization**
- **Generalized flow :**
  Ensembles of random permutations

$$t, \mathbf{i_0} \mapsto \boldsymbol{\sigma}_t^{\omega}(i_0)$$




## Monte-Carlo estimates

- **Gibbs** measure (with BC) : $p_\beta = \dfrac{1}{Z(\beta)} e^{-\beta \mathcal{A}_d[\boldsymbol{\sigma}]}$
- **Discrete Action :** $\mathcal{A}_d[\boldsymbol{\sigma}] = \sum_{n=1}^{N_t} \sum_{\mathbf{i}} \|\boldsymbol{\sigma}_n(\boldsymbol{i}) - \boldsymbol{\sigma}_{n-1}(\boldsymbol{i})\|^2$

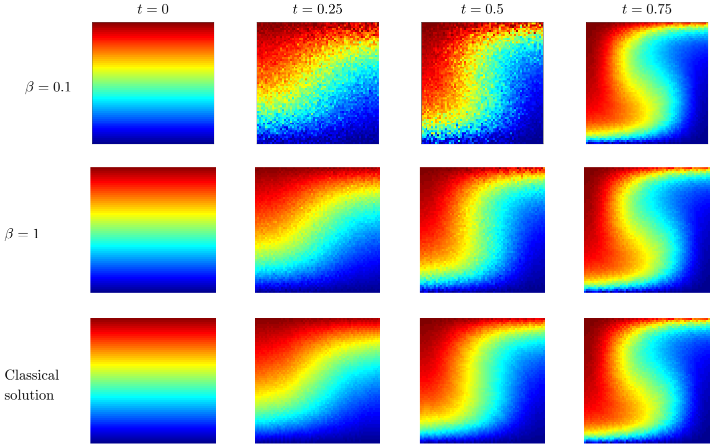
*Remark :* Finite $\beta$ fluctuations akin to the "entropic regularization" used in NENNA, 2016 (PHD) ; BENAMOU & AL, 2015

# Layout

## Small $t_f$ : Reconstruction of a classical solution *(i)*

*Test case :* Stationary cellular velocity field on $\Omega = [0,\pi]^2$ (Beltrami):

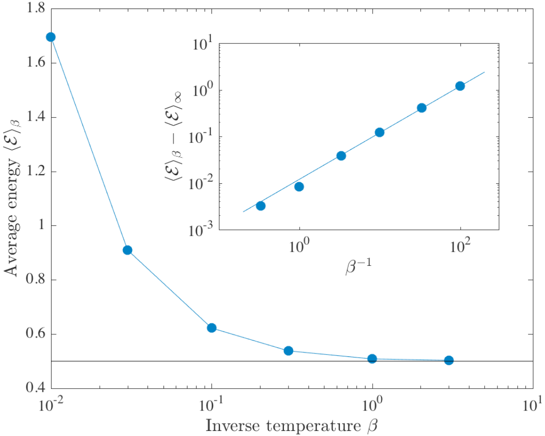
$$v_B(\mathbf{x},t) = \pi(-\cos y \sin x\,\hat{\mathbf{x}} + \cos x \sin y\,\hat{\mathbf{y}}) \qquad \text{Critical time : } t_f^\star = 1$$

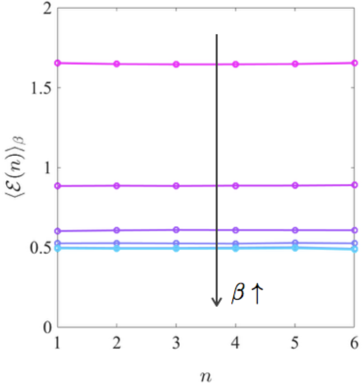



Coarse-graining on a grid with size $N_x^2 = 64^2$

Timesteps of size $\Delta t = t_f^\star/8$

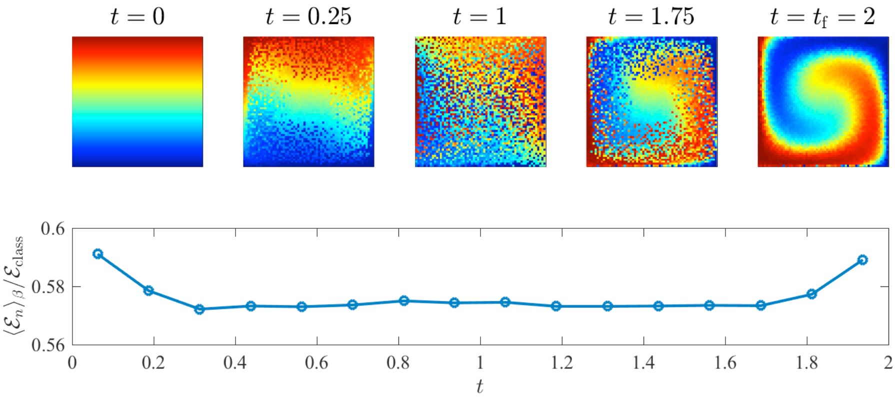
# Small $t_f$ : Reconstruction of a classical solution *(ii)*



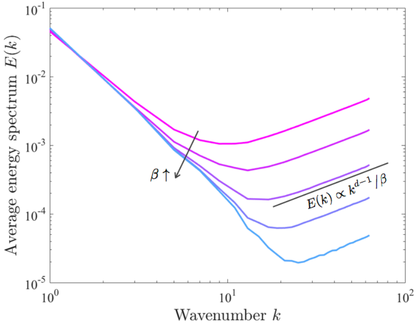

$\Longrightarrow$ Convergence towards classical solution in the zero temperature limit $\beta \to \infty$

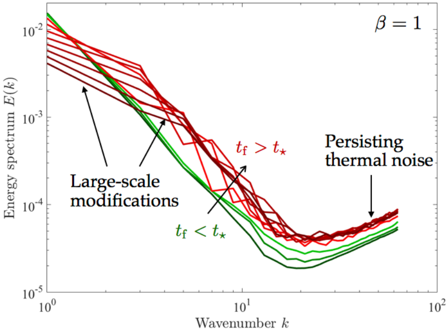
# Large $t_f$ : Non-deterministic solution




- The intermediate generalized dynamics is **not** Beltrami
- Is it "Thermalized" ? or "Turbulent" ? or even "Physical"?

# Eulerian features of the generalized Beltrami flows



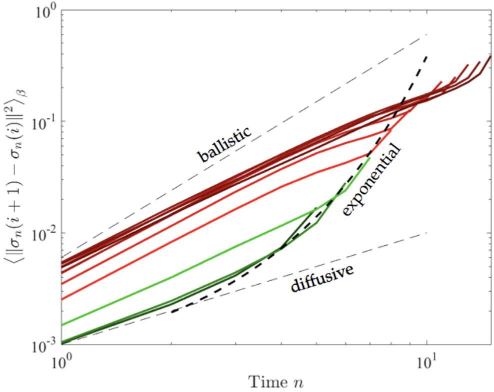

Small $t_f$ : Convergence of the spectra as $\beta \to \infty$.

Various $t_f$: Spectra at $t_f/2$.

$\Longrightarrow$ Generalized flows for large $t_f$ have non-trivial IR signatures, different from random flows.

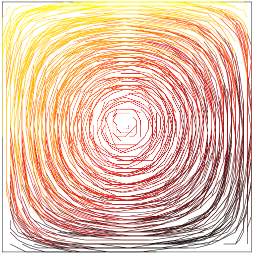
# Lagrangian features of the generalized Beltrami flows

## Growth of separations

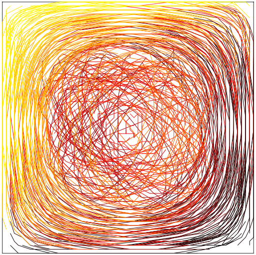



$\Longrightarrow$ Lagrangian statistics are not “turbulent”.

## Lagangian trajectories



Small $t_f$

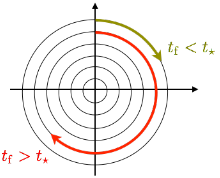


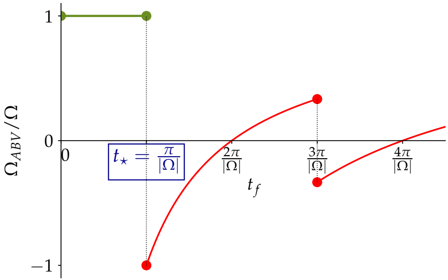
Large $t_f$

# Variationnal principle and large final times

- The maximal final time is determined as : $t_f^\star = \dfrac{\pi}{\sup_{x,t} \| \mathit{Hessian}[p] \|^{1/2}}$.
- "Defect" of the Boundary-value formulation itself
- **Illustrative example :**

Reconstruction of a solid-rotation pulse $\Omega$ from Arnold's principle.

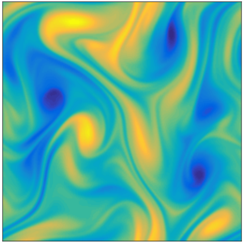


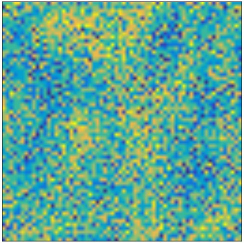

$\Rightarrow$ Shortcuts are cheaper for large final time!

# Reconstruction of a non-stationary dynamics : decaying 2D

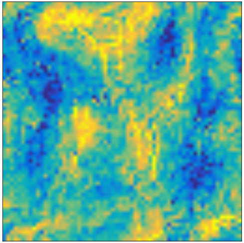
$\omega(\mathbf{x}, t)$

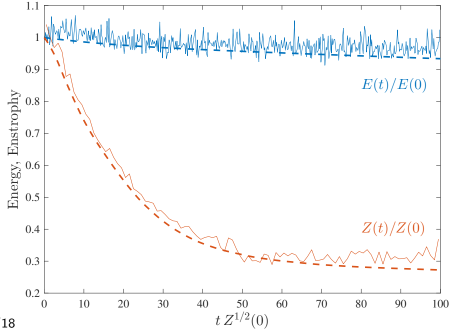


$\bar{\omega}(\mathbf{x}, t)$



$\langle \bar{\omega}(\mathbf{x}, t) \rangle_\beta$






In the limit $\nu \to 0$, dynamics is in principle described by **irreversible weak** solutions of the Euler equation.
(e.g. Eyink , 2001)

$t_f \sim Z_0^{-1/2}$

Original simulation : $1024^2$

Generalized flow : $64^2$

$\Rightarrow$ Irreversibility encoded in the final map?

# Final messages

## Conclusions

- Boundary value formulation is ill posed for large timelags $\Rightarrow$ the Corresponding generalized flows are then unphysical.
- For small timelags generalised flows can capture irreversible behaviours
- Possible tool to coarse-grain turbulent flows...
- ... provided some weak Euler solutions are themselves relevant for turbulence.

## Perspectives/Work in progress

- Reconstruction of multiscale turbulent measures ?
  *e.g.* 2D Inverse cascade/ 3D direct cascade
- Beyond MC algorithm ? Semi-discrete transport, Entropic Regularization...
  Mérigot & Mirebeau, 2015 ; Nenna, 2016, ...
- Beyond $t_\star$ : Further constraints (Energy/Enstrophy)?
  Generalised Conservation laws?