Hidden Symmetries in Navier-Stokes intermittency.

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1. **Classical symmetries**
2. Navier-Stokes under dynamical rescaling
3. Hidden symmetries

Navier-Stokes in $\mathbb{R}^3$

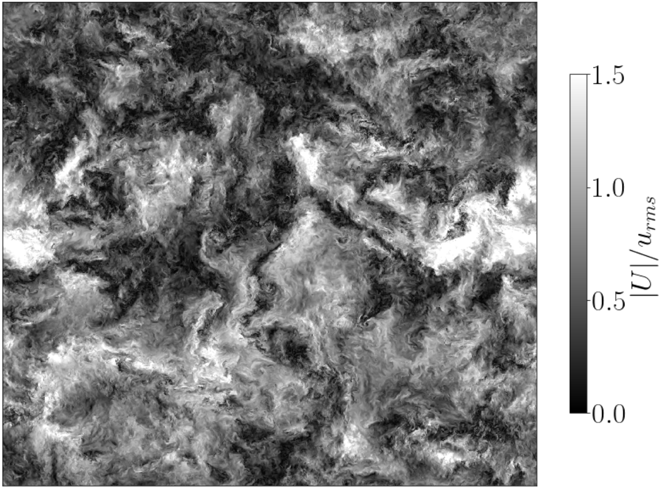
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \mathbf{0}.$$

Symmetries

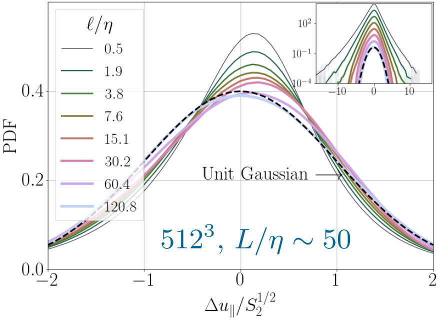
	parameters	$t \mapsto$	$\mathbf{x} \mapsto$	$\mathbf{u} \mapsto$	$\nu \mapsto$	$\mathbf{f} \mapsto$
Galilean	$\mathbf{u}_0 \in \mathbb{R}^3$	t	$\mathbf{x} + t\mathbf{u}_0$	$\mathbf{u} + \mathbf{u}_0$	ν	$\mathbf{f}$
Translation	$\Delta t \in \mathbb{R},\ \Delta \mathbf{x} \in \mathbb{R}^3$	$t + \Delta t$	$\mathbf{x} + \Delta \mathbf{x}$	$\mathbf{u}$	ν	$\mathbf{f}$
Rotation	$\mathbf{O} \in \mathrm{SO}(3)$	t	$\mathbf{Ox}$	$\mathbf{Ou}$	ν	$\mathbf{Of}$
Scaling	$h, \lambda > 0$	$\lambda^{1-h} t$	$\lambda \mathbf{x}$	$\lambda^h \mathbf{u}$	$\lambda^{1+h} \nu$	$\lambda^{1+2h} \mathbf{f}$

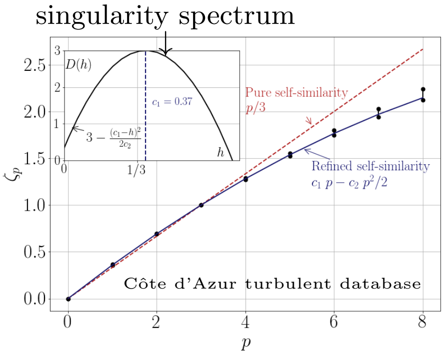
e.g.

$$\mathbf{u}'(\mathbf{x}', t') := \lambda^h \mathbf{u}\left(\frac{\mathbf{x}'}{\lambda}, \frac{t'}{\lambda^{1-h}}\right) \quad \text{solves} \quad \partial_{t'} \mathbf{u}' + \mathbf{u}' \cdot \nabla' \mathbf{u}' + \nabla' p' = \nu' \Delta' \mathbf{u}' + \mathbf{f}', \quad \nabla' \cdot \mathbf{u}' = 0.$$



Statistical symmetries: Rotation : ✓ Translation : ✓ Scaling : ×





Multi-fractals: Range of scaling exponents $h \in (h_{\min}, h_{\max})$

$$\left\langle \Delta \mathbf{u}_{\parallel}^{p} \right\rangle = \int_{h_{\min}}^{h_{\max}} \mathrm{d}\overset{\propto \ell^{3-D(h)}}{\overset{\downarrow}{\mu}}(h) \left\langle \Delta \mathbf{u}_{\parallel}^{p} \right\rangle_{\mathcal{S}_h} \propto \ell^{\zeta_p}$$

Extrinsic: Scale-invariance of the normalized increments

$$\frac{u_k(\mathbf{x}+\ell\mathbf{X})-u_k(\mathbf{x},t)}{\ell^{1/3}\epsilon_\ell^{1/3}(\mathbf{x})}, \quad \epsilon_\ell(\mathbf{x}) := \left\langle \nu\|\nabla\mathbf{u}\|^2\right\rangle_{B(\mathbf{x},\ell)} \quad \text{(local dissipation)}$$

Intrinsic: Scale-invariance of the multipliers

$$w_{ij,k}(\mathbf{x},t;\ell_1,\ell_2) := \frac{u_k(\mathbf{x}+\ell_1\mathbf{e}_i,t)-u_k(\mathbf{x},t)}{u_k(\mathbf{x}+\ell_2\mathbf{e}_j,t)-u_k(\mathbf{x},t)}.$$

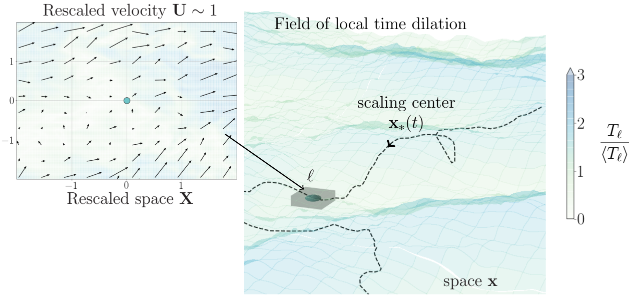
Can one explicitly relate refined self-similarity to non-degenerate symmetries?

Layout

1. Classical symmetries

2. **Navier-Stokes under dynamical rescaling**

3. Hidden symmetries



$t, \mathbf{x}, \mathbf{u} \mapsto \tau, \mathbf{X}, \mathbf{U}$

1. Space rescaling in **quasi-Lagrangian** frame

$$\mathbf{U}(\mathbf{X}, \tau; \mathbf{x}_0, \ell) := \frac{\Delta \mathbf{u}_\ell(\mathbf{x}_*(t; \mathbf{x}_0), \mathbf{X}, t)}{A_\ell(x_*(t), t)}$$

2. **Proper time** $d\tau := \dfrac{dt}{T_\ell(x_*(t), t)}, \qquad T_\ell(\mathbf{x}, t) := \dfrac{\ell}{A_\ell(\mathbf{x}, t)}$

Consider inertial range dynamics

$$\ell \ll 1 \qquad \ell \gg \frac{\nu}{A_\ell(\mathbf{x}_*, t)} \quad \text{(“local Kolmogorov scale”)}$$

Then the rescaled dynamics become the rescaled Euler system

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P\right] = 0, \quad \nabla \cdot \mathbf{U} = 0,$$

where

$$\Lambda_{\mathbf{U}}[\mathbf{V}] = \mathbf{V} - \mathbf{V}(0) - \mathbf{U}(\cdot, \tau) \left(\left.\frac{\delta \mathcal{A}}{\delta \mathbf{V}}\right|_{\mathbf{U}(\cdot,\tau)} \cdot (\mathbf{V} - \mathbf{V}(0)) \right)$$

1. Classical symmetries
2. Navier-Stokes under dynamical rescaling
3. **Hidden symmetries**

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} [\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P] = 0, \quad \nabla \cdot \mathbf{U} = 0,$$

Explicit symmetries

	parameters	$\tau \mapsto$	$\mathbf{X} \mapsto$	$\mathbf{U} \mapsto$
Rotation	$\mathbf{O} \in \mathrm{SO}(3)$	τ	$\mathbf{OX}$	$\mathbf{OU}$
Time translation	$\Delta\tau \in \mathbb{R}$	$\tau + \Delta\tau$	$\mathbf{X}$	$\mathbf{U}$

"Hidden" symmetries

	parameters	$\tau \mapsto$	$\mathbf{X} \mapsto$	$\mathbf{U} \mapsto$
Hidden translation	$\mathbf{X}_0 \in \mathbb{R}^3$	$\widetilde{\tau}$	$\mathbf{X}$	$\widetilde{\mathbf{U}}$
Hidden scaling	$\lambda > 0$	τ'	$\mathbf{X}$	$\mathbf{U}'$

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P\right] = 0, \quad \nabla \cdot \mathbf{U} = 0,$$

Observation

The rescaled Euler dynamics for $\mathbf{U}(\mathbf{X}, \tau; \ell, \mathbf{x}_0)$ is invariant under the change of averaging scale $\ell \mapsto \ell/\lambda$.

Proposition

Under the change $\ell \mapsto \ell/\lambda$, the proper variables transform as

$$\tau, \mathbf{X}, \mathbf{U} \mapsto \tau', \mathbf{X}, \mathbf{U}'$$

$$\text{for} \quad \mathbf{U}'(\mathbf{X}, \tau') := \frac{\mathbf{U}_\lambda(\mathbf{X}, \tau)}{\mathcal{A}[\mathbf{U}_\lambda(\cdot, \tau)]}, \quad \tau' := \lambda \int_0^\tau \mathcal{A}[\mathbf{U}_\lambda(\cdot, s)]\, ds$$

$$\text{where} \quad \mathbf{U}_\lambda(\mathbf{X}, \tau) := \mathbf{U}\left(\frac{\mathbf{X}}{\lambda}, \tau\right)$$

Symmetries for original Euler

$$t, \mathbf{x}, \mathbf{u}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$$

	parameters	$t \mapsto$	$\mathbf{x} \mapsto$	$\mathbf{u} \mapsto$
Rotation	$\mathbf{O} \in \mathrm{SO}(3)$	t	$\mathbf{O}\mathbf{x}$	$\mathbf{O}\mathbf{u}$
Time translation	$\Delta t \in \mathbb{R}$,	$t + \Delta t$	$\mathbf{x}$	$\mathbf{u}$
Galilean	$\mathbf{u}_0 \in \mathbb{R}^3$	t	$\mathbf{x} + t\mathbf{u}_0$	$\mathbf{u} + \mathbf{u}_0$
Space translation	$\Delta \mathbf{x} \in \mathbb{R}^3$	t	$\mathbf{x} + \Delta \mathbf{x}$	$\mathbf{u}$
Scaling	$h, \lambda > 0$	$\lambda^{1-h} t$	$\lambda \mathbf{x}$	$\lambda^h \mathbf{u}$

8 parameters

Symmetries for rescaled Euler

$$\tau, \mathbf{X}, \mathbf{U}$$

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P \right] = 0$$

	parameters	$\tau \mapsto$	$\mathbf{X} \mapsto$	$\mathbf{U} \mapsto$
Rotation	$\mathbf{O} \in \mathrm{SO}(3)$	τ	$\mathbf{O}\mathbf{X}$	$\mathbf{O}\mathbf{U}$
Time translation	$\Delta \tau \in \mathbb{R}$	$\tau + \Delta \tau$	$\mathbf{X}$	$\mathbf{U}$
Hidden translation	$\mathbf{X}_0 \in \mathbb{R}^3$	$\widetilde{\tau}$	$\mathbf{X}$	$\widetilde{\mathbf{U}}$
Hidden scaling	$\lambda > 0$	τ'	$\mathbf{X}$	$\mathbf{U}'$

4 parameters

Proposition: Hidden scaling fuses multi-fractal scaling

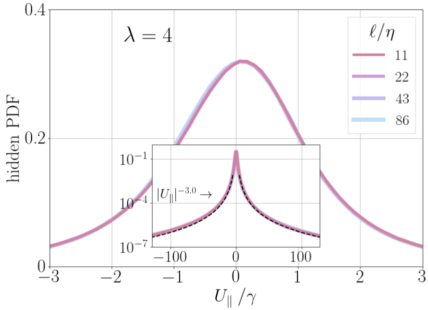
$$\begin{array}{ccc}
t, \mathbf{x}, \mathbf{u} & \longrightarrow & \lambda^{1-h}t, \lambda\mathbf{x}, \lambda^{h}\mathbf{u} \\
{\scriptstyle \mathbf{x}_0,\ \ell}\big\downarrow & & \big\downarrow{\scriptstyle \lambda\mathbf{x}_0,\ \ell} \\
\tau, \mathbf{X}, \mathbf{U} & \xrightarrow{\ \ell \mapsto \ell/\lambda\ } & \tau', \mathbf{X}, \mathbf{U}'
\end{array}$$

Implication for turbulence (Conjecture)

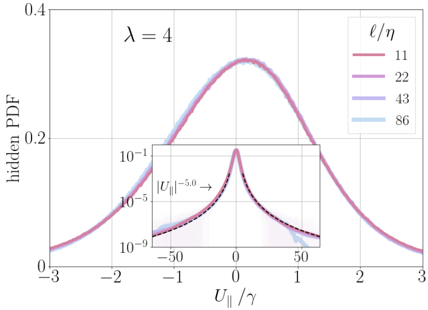
In the inertial range, hidden scaling symmetry may hold although scaling symmetry in the usual sense is broken.

Numerical tests : $4,096^3$, $L/\eta \sim 450$

$$\mathcal{A}[\mathbf{V}] = |V_\parallel(\mathbf{e})|$$

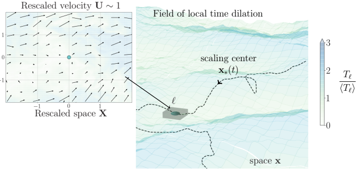


$$\mathcal{A}[\mathbf{V}] = \left|\left\langle \|\mathbf{V}\|^2 V_\parallel \right\rangle_S\right|^{1/3}$$



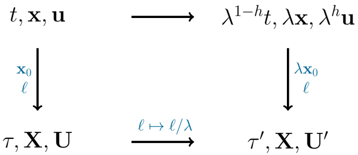
Côte d'Azur Turbulent Database

1. Dynamical rescaling

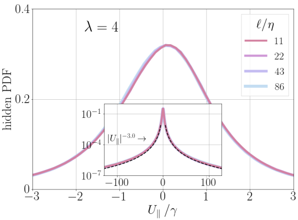


$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P \right] = 0$$

2. Fusing old into hidden symmetries



3. Statistical hidden scaling symmetry



Extended story:

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