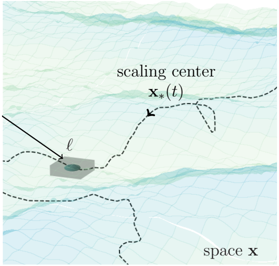
Turbulent Roads to intermittency:

From zero-modes to hidden symmetry

Simon Thalabard



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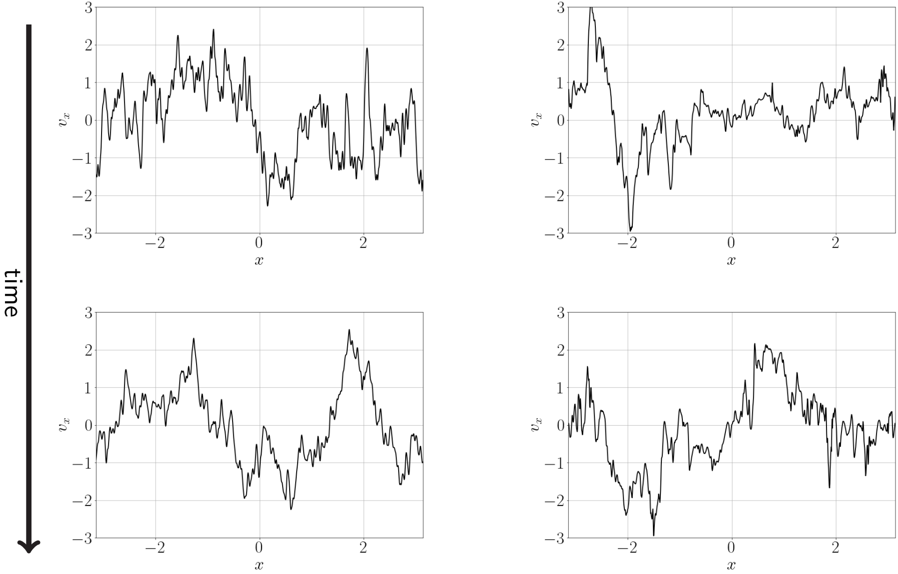
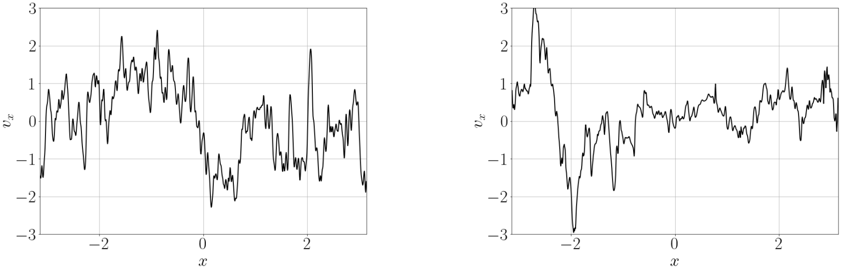
1. **Intermittency**

Turbulent fluctuations



time

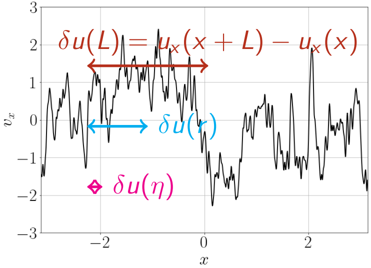


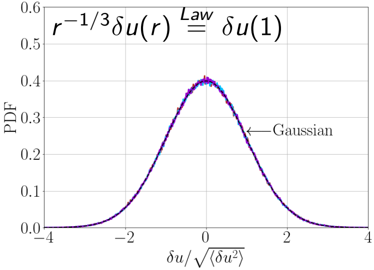
(i) Stationnary

(ii) Finite variance $\quad \langle u_x^2 \rangle = O(1)$

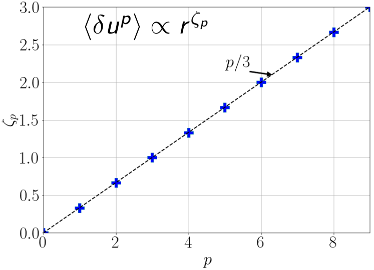
(iii) Power-law correlations $\quad \langle u_x(x) u_x(x+r) \rangle \propto 1 - r^{\xi}$

Gaussian turbulence

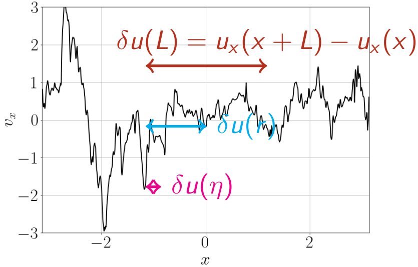


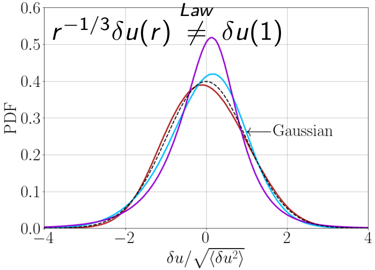


Scale invariance

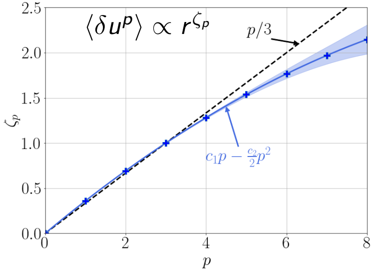


Monofractal scaling



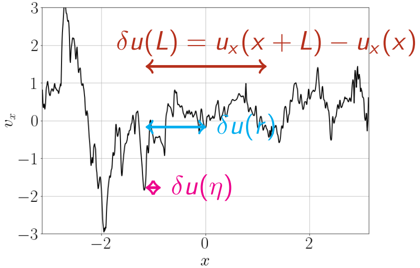


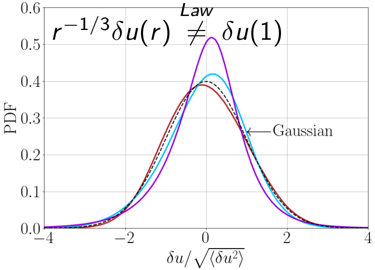
No obvious scale invariance

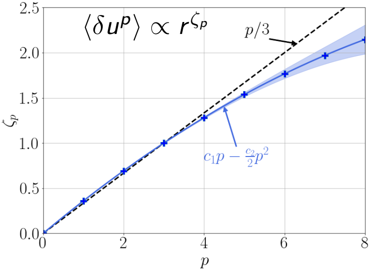


Anomalous scaling

Intermittency :
Deviations from Gaussian turbulence

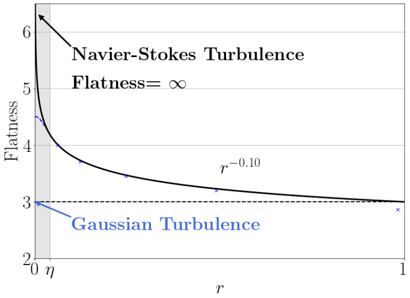


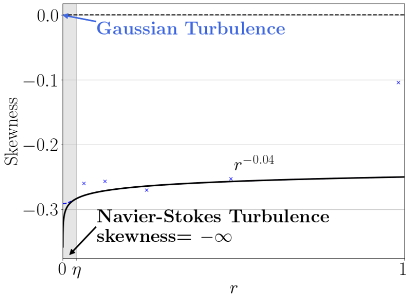




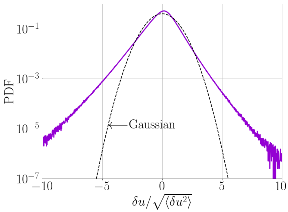
No obvious scale invariance

Anomalous scaling

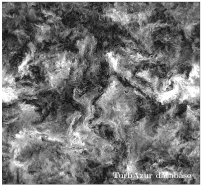


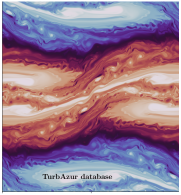


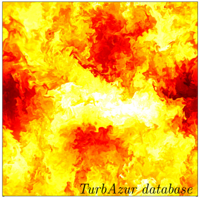
$$\zeta_4 - 2\zeta_2 \simeq= -0.10 \qquad \zeta_3 - \frac{3}{2}\zeta_2 \simeq -0.04$$



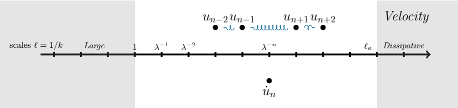
Navier-Stokes	Active scalar	Passive	Gaussian advection

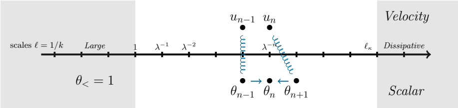






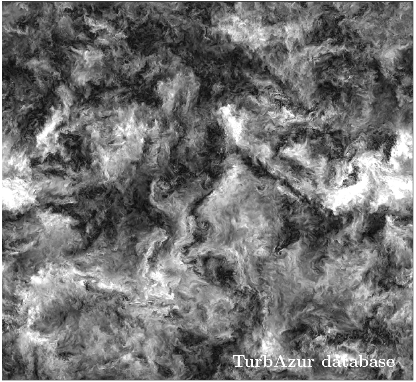
Non-linear | **Linear**





Cascade models

Sabra, GOY, Dyadic,... | Sabra/Gaussian advection

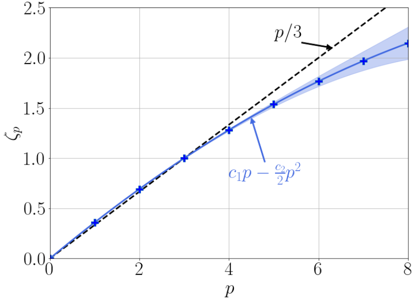


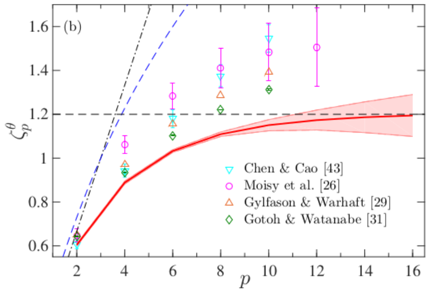
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \kappa \Delta \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \mathbf{f}_\theta + \kappa \Delta \theta$$

Navier-Stokes vs Passive scalar Intermittency





Iyer & al, PRL ('18)

$$\langle |\delta u|^p \rangle \propto \ell^{\zeta_p}$$

$$\langle |\delta \theta|^p \rangle \propto \ell^{\zeta_p^\theta}$$

- Why scaling ?
- Why anomalous?

2 dynamical mechanisms for intermittency

Conservation laws

- Exact laws
- **Zero modes**
  Kraichnan flow Theory
  O('90 - '00)

Symmetries

- Refined self-similarity
  Kolmogorov ('61)
- Multifractal Parisi-Frisch ('85)
- **Hidden symmetries**
  Mailybaev ('20)

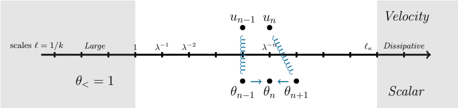
Layout

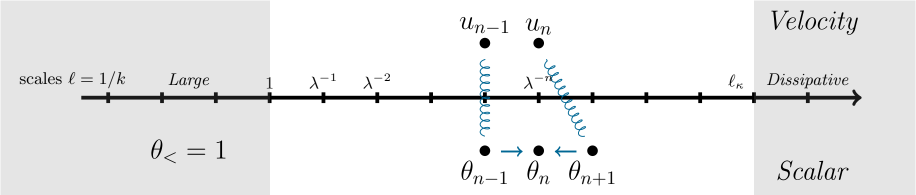
1. Intermittency
2. **Conservation laws (Zero-modes)**

Where intermittency?

Non-linear

Linear



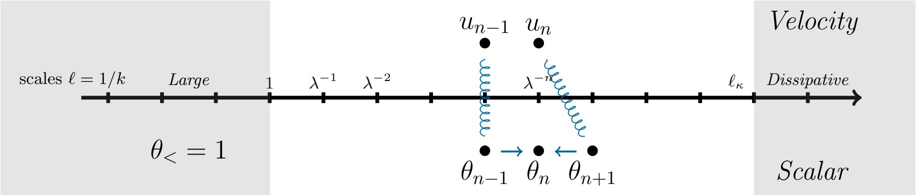


Kraichnan-Wirth-Biferale (KWB) model

Jensen & al ('92), Biferale & Wirth ('96 '07), Andersen & Muratore-Ginanneschi ('99)

$$\dot{\theta}_n = k_n \theta_{n+1} u_n - k_{n-1}\theta_{n-1}u_{n-1} - \kappa k_n^2 \theta_n$$

$$u \sim \text{Gaussian} \quad \text{with } \langle u_n(t) u_m(t+\tau)\rangle \propto \ell_n^{2/3}\, \delta_{nm}\, \delta(\tau)$$



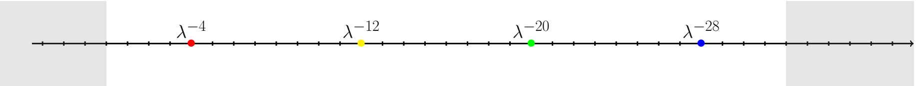
Inertial steady-state $1 \gg \ell_n \gg \ell_\kappa \propto \kappa^{-3/2}$

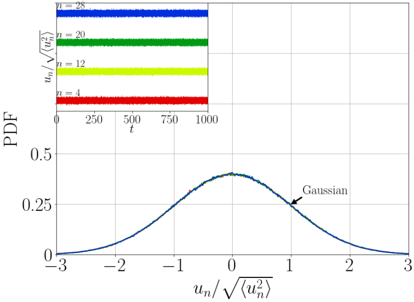
- Kolmogorov scaling

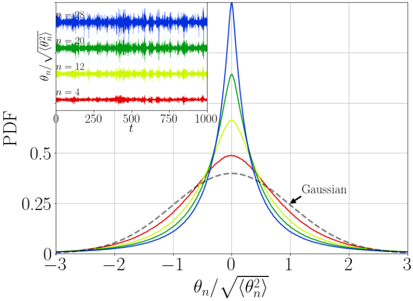
$$\langle \theta_n^2 \rangle \propto \ell_n^{4/3}$$

- Constant scalar flux

$$1 = \Pi_n = \ell_n^{-4/3} \langle \theta_n^2 \rangle - \ell_n^{-4/3} \langle \theta_{n+1}^2 \rangle$$



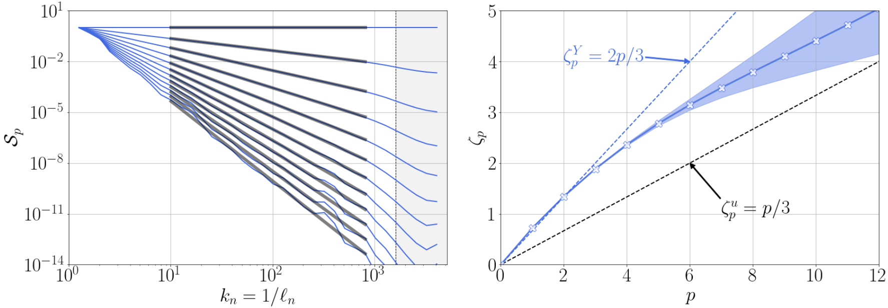




Gaussian velocity $\rightsquigarrow$ Non-Gaussian scalar!

Numerical observations: Structure functions

$$\mathcal{S}_p(\ell_n) := \lim_{T\to\infty} T^{-1}\int_0^T \langle|\theta_n|^p\rangle\, dt \quad \propto \ell_n^{\zeta_p}$$



Minimal model for (random) scalar intermittency!

$$\mathcal{S}_2 := \lim_{T\to\infty} T^{-1}\int_0^T \langle|\theta_n|^2\rangle\, dt$$

Inertial range recursion

$$0 = \mathcal{S}_2(\ell_{n-1}) - (1+\gamma^2)\mathcal{S}_2(\ell_n) + \gamma^2 S_2(\ell_{n+1}), \qquad \gamma := \lambda^{2/3}$$

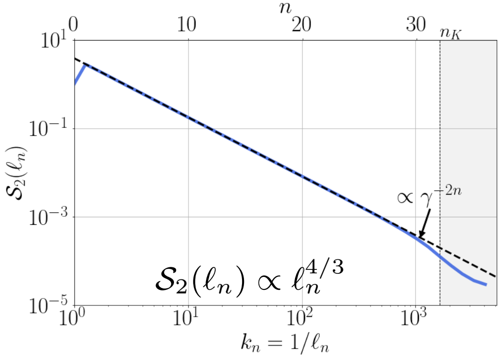
$$\text{with boundary conditions} \quad \begin{cases} \mathcal{S}_2(\ell_n) \underset{\infty}{\to} 0 \text{ (small-scale)} \\ \mathcal{S}_2(\ell_0) = 1 \text{(large scale)}. \end{cases}$$

Zero-mode interpretation

$$\underbrace{\begin{pmatrix} \ddots & & \ddots & & \\ \ddots & -(1+\gamma^2) & \gamma^2 & & \\ & 1 & -(1+\gamma^2) & \ddots \\ & & \ddots & \ddots \end{pmatrix}}_{M_2} \mathcal{S}_2 = 0$$

$$\mathcal{S}_2 := \lim_{T\to\infty} T^{-1} \int_0^T \langle |\theta_n|^2 \rangle \, dt$$

Kolmogorov scaling is a (trivial) zero-mode



$$\mathcal{S}_4(\ell_n) := \sigma_{n0}, \qquad \sigma_{nl} = \lim_{T\to\infty} T^{-1} \int_0^T \left\langle \theta_n^2 \theta_{n+l}^2 \right\rangle dt.$$

Inertial range recursion

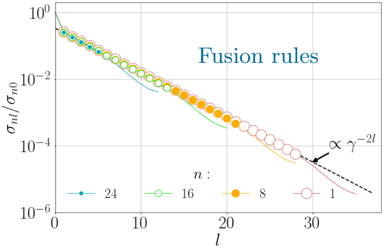
$$\mathcal{M}_4\sigma = 0 \qquad \begin{aligned} 0 = \quad & b_{-l}\sigma_{(n+1)(l-1)} \\ & + b_l\gamma^{-2}\sigma_{n(l-1)} - a_l\sigma_{nl} + b_l\sigma_{n(l+1)} \\ & + b_{-l}\gamma^{-2}\sigma_{(n-1)(l+1)} \end{aligned} \quad \text{for} \begin{cases} a_l = \gamma^l + \gamma^{-l-2} \\ \quad + \gamma^{-l} + \gamma^{l-2} + 4\gamma^{-l}\delta_{l1}, \\ b_l = \gamma^l + 2\delta_{l0}. \end{cases}$$

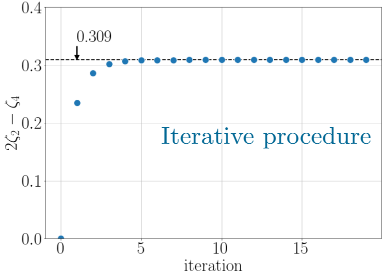
Benzi Ansatz Benzi & al ('97)

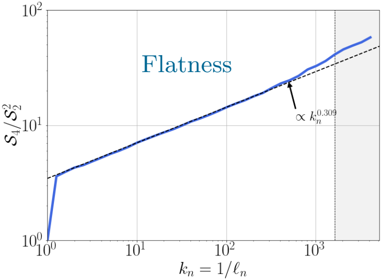
- $\mathcal{S}_4(\ell_n) = \sigma_{n0} \propto \ell_n^{\zeta_4}$ (Scaling Ansatz)
- $\sigma_{nl} = C_l\sigma_{n0}$ with $C_l \underset{\infty}{\to} 0$ & $C_0 = 1$ (Fusion Rule)

The Benzi Ansatz yields the fixed point:

$$\zeta_4 = \log_\lambda \left(\frac{1+\gamma^2}{3C_1(\zeta_4)} - \gamma^2 \right).$$







Statistical conservation laws

The ideal KWB dynamics preserve

$$\Gamma_2 := \sum_{n\in\mathbb{Z}} \gamma^{-2n} \left\langle \theta_n^2 \right\rangle, \qquad \Gamma_4 = \sum_{n\in\mathbb{Z}} \sum_{l\geq 0} c_l r^n \left\langle \theta_n^2 \theta_{n+l}^2 \right\rangle,$$

where $c_l = 6C_l - 5\delta_{l0}$ and $r = \lambda^{-\zeta_4}$.

Explicit duality

$$\mathcal{M}_2 \mathcal{S}_2 = 0 \iff \mathcal{M}_2^\dagger \begin{pmatrix} \vdots \\ \gamma^{-2n} \\ \vdots \end{pmatrix} = 0, \qquad \mathcal{M}_4 \sigma = 0 \iff \mathcal{M}_4^\dagger \begin{pmatrix} \vdots \\ c_l r^n \\ \vdots \end{pmatrix} = 0.$$

- Hierarchy of non-trivial conservation laws $\Gamma_2, \Gamma_4, \Gamma_6 \ldots$
  Beyond p=4 and white-in-time setting: ANDERSEN & MURATORE-GINANNESCHI ('99)

- Duality between zero modes and statistical conservation laws.
  Beyond linear setting: ANGHELUTA & AL ('06), ARAD & AL ('01)

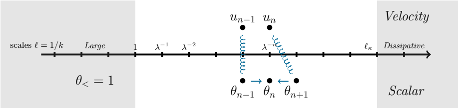
- Even-order exponents $\zeta_0, \zeta_2, \zeta_4,\ \zeta_6\ \ldots$

Scope of Zero-Mode theory

Lagrangian conservation laws

Non-linear

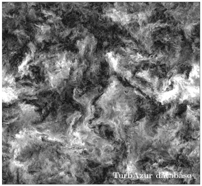
Linear



Statistical conservation laws

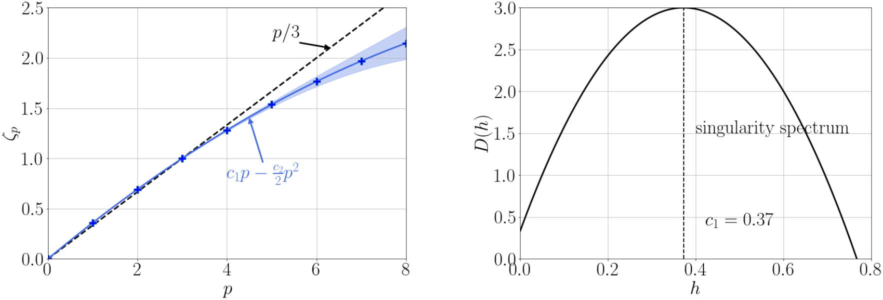
1. Intermittency
2. Conservation laws (Zero-modes)
3. **(Hidden) Symmetries**

Where intermittency ?



Non-linear

Linear



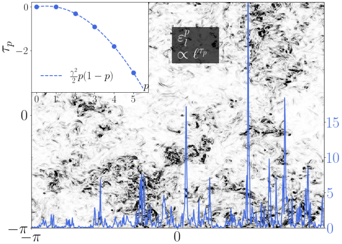
Range of scaling exponents $h \in (h_{\min}, h_{\max})$

$$\left\langle \Delta \mathbf{u}_{\|}^{p} \right\rangle = \int_{h_{\min}}^{h_{\max}} \mathrm{d}\overset{\propto \ell^{3-D(h)}}{\overset{\downarrow}{\mu}}(h) \left\langle \Delta \mathbf{u}_{\|}^{p} \right\rangle_{\mathcal{S}_h} \propto \ell^{\zeta_p}, \qquad \zeta_p = \inf\left\{3 - D(h) + ph\right\}$$

Entangled scaling symmetries!

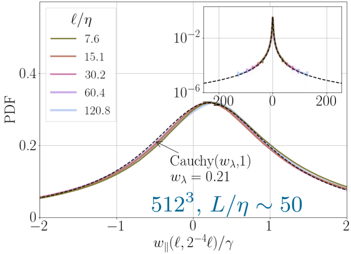
(i) Conditionning on local dissipation

$$\frac{u_k(\mathbf{x}+\ell\mathbf{X})-u_k(\mathbf{x},t)}{\ell^{1/3}\epsilon_\ell^{1/3}(\mathbf{x})}, \quad \epsilon_\ell(\mathbf{x}) := \left\langle \nu\|\nabla\mathbf{u}\|^2\right\rangle_{B(\mathbf{x},\ell)}$$



(ii) Multipliers

$$w_{ij,k}(\mathbf{x},t;\ell_1,\ell_2) := \frac{u_k(\mathbf{x}+\ell_1\mathbf{e}_i,t)-u_k(\mathbf{x},t)}{u_k(\mathbf{x}+\ell_2\mathbf{e}_j,t)-u_k(\mathbf{x},t)}.$$



Random-h phenomenology

- Scale: $\ell_n = 2^{-n}$
- Identity: $\delta u(\ell_n) = \delta u(1) \times \frac{\delta u(2)}{\delta u(1)} \times \cdots \times \overbrace{\frac{\delta u(\ell_n)}{\delta u(\ell_{n-1})}}^{w_n}$
- Structure function: $\langle \delta u^p(\ell_n) \rangle = \left\langle \delta u^p(1) \prod_{i=1}^{n} w_i^p \right\rangle$

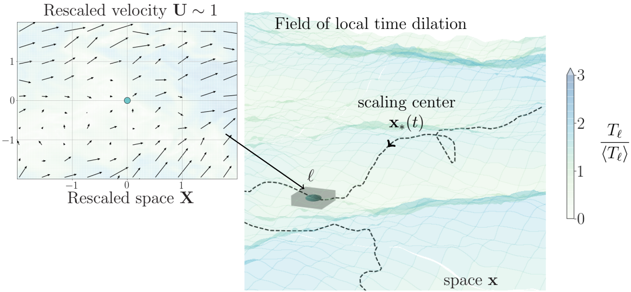
Multifractal examples: $w_i = 2^{-h_i}$

1. $h_i \sim \mathcal{N}\left(c_1, \frac{c_2}{\log 2}\right)$ iid $\quad \Longrightarrow \quad \langle \delta u^p(\ell_n) \rangle \propto \ell_n^{\zeta_p}, \quad \zeta_p = c_1 p - \frac{p^2}{2} c_2.$
2. $H = \frac{1}{N} \sum_{i=1}^{N} h_i \sim 2^{ND(h)} \quad \Longrightarrow \quad \zeta_p = \inf_h \{ph - D(h)\}$

Universality of multipliers from the Navier-Stokes?

Dynamical (Hidden) rescaling

$t, \mathbf{x}, \mathbf{u} \mapsto \tau, \mathbf{X}, \mathbf{U}$



1. Local averaging $\qquad A_\ell(\mathbf{x}, t) := \left\langle \|\Delta \mathbf{u}_\ell\|^2 \right\rangle^{1/2}_{\|\mathbf{X}\|=1}$

2. **Quasi-Lagrangian** frame $\qquad \mathbf{U}(\mathbf{X}, \tau; \mathbf{x}_0, \ell) := \dfrac{\Delta \mathbf{u}_\ell(\mathbf{x}_*(\mathbf{t}), \mathbf{X}, \mathbf{t})}{A_\ell(x_*(t), t)}$

3. **Proper time** $\qquad \mathrm{d}\tau := \dfrac{A_\ell(x_*(t), t)}{\ell} dt$

$\mathbf{U}(\mathbf{X},\tau;\ell,\mathbf{x}_0)$

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}}\left[\mathbf{U}\cdot\nabla\mathbf{U} + \nabla\mathcal{P}\right] = \Lambda_{\mathbf{U}}\left[\nu_\ell \Delta\mathbf{U} + \mathcal{F}_\ell\right],$$

$$\nabla\cdot\mathbf{U} = 0,$$

$$\Lambda_{\mathbf{U}}[\mathbf{V}] = \mathbf{V} - \mathbf{U}\left\langle\mathbf{U}\cdot\mathbf{V}\right\rangle_{|\mathbf{X}|=1}$$

Hidden inertial range

$$1 \gg \quad \ell \quad \gg \quad \frac{\nu}{A_\ell(\mathbf{x}_*,t)} \qquad \text{(local Kolmogorov scale)}$$

$\mathbf{U}(\mathbf{X}, \tau; \ell, \mathbf{x}_0)$

$$\boxed{\begin{gathered} \partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathcal{P} \right] = 0, \\ \nabla \cdot \mathbf{U} = 0, \\ \Lambda_{\mathbf{U}}[\mathbf{V}] = \mathbf{V} - \mathbf{U} \left\langle \mathbf{U} \cdot \mathbf{V} \right\rangle_{|\mathbf{X}|=1} \end{gathered}}$$

Hidden scale invariance

Invariance under the change of averaging scale

$$\ell \mapsto \ell/\lambda : \qquad \tau, \mathbf{X}, \mathbf{U} \mapsto \tau', \mathbf{X}, \mathbf{U}'$$

Hidden translation

Invariance under the change of reference trajectory $\mathbf{x_0} \mapsto \tilde{\mathbf{x}}_0$:

$$\mathbf{x_0} \mapsto \tilde{\mathbf{x}}_0 : \qquad \tau, \mathbf{X}, \mathbf{U} \mapsto \tilde{\tau}, \mathbf{X}, \tilde{\mathbf{U}}$$

Symmetries for original Euler

$$t, \mathbf{x}, \mathbf{u}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$$

	parameters	$t \mapsto$	$\mathbf{x} \mapsto$	$\mathbf{u} \mapsto$
Rotation	$\mathbf{O} \in SO(3)$	t	$\mathbf{Ox}$	$\mathbf{Ou}$
Time translation	$\Delta t \in \mathbb{R}$,	$t + \Delta t$	$\mathbf{x}$	$\mathbf{u}$
Galilean	$\mathbf{u}_0 \in \mathbb{R}^3$	t	$\mathbf{x} + t\mathbf{u}_0$	$\mathbf{u} + \mathbf{u}_0$
Space translation	$\Delta \mathbf{x} \in \mathbb{R}^3$	t	$\mathbf{x} + \Delta \mathbf{x}$	$\mathbf{u}$
Scaling	$h, \lambda > 0$	$\lambda^{1-h} t$	$\lambda \mathbf{x}$	$\lambda^h \mathbf{u}$

8 parameters

Symmetries for Hidden Euler

$$\tau, \mathbf{X}, \mathbf{U}$$

$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P \right] = 0$$

	parameters	$\tau \mapsto$	$\mathbf{X} \mapsto$	$\mathbf{U} \mapsto$
Rotation	$\mathbf{O} \in SO(3)$	τ	$\mathbf{OX}$	$\mathbf{OU}$
Time translation	$\Delta \tau \in \mathbb{R}$	$\tau + \Delta \tau$	$\mathbf{X}$	$\mathbf{U}$
Hidden translation	$\mathbf{X}_0 \in \mathbb{R}^3$	$\widetilde{\tau}$	$\mathbf{X}$	$\widetilde{\mathbf{U}}$
Hidden scaling	$\lambda > 0$	τ'	$\mathbf{X}$	$\mathbf{U}'$

4 parameters

Hidden scaling fuses multi-fractal scaling

$$\begin{tikzcd}
t, \mathbf{x}, \mathbf{u} \arrow[r] \arrow[d, "\mathbf{x}_0" ', "\ell" '] & \lambda^{1-h}t, \lambda\mathbf{x}, \lambda^h\mathbf{u} \arrow[d, "\lambda\mathbf{x}_0", "\ell"] \\
\tau, \mathbf{X}, \mathbf{U} \arrow[r, "\ell \mapsto \ell/\lambda"] & \tau', \mathbf{X}, \mathbf{U}'
\end{tikzcd}$$

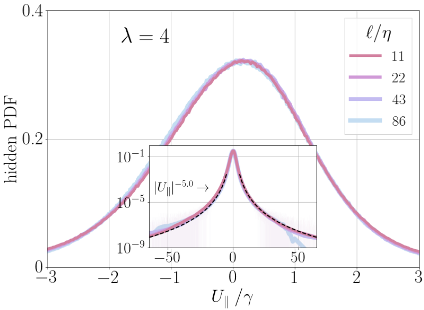
Hidden translation fuses translation and galilean invariance

$$\begin{tikzcd}
t, \mathbf{x}, \mathbf{u} \arrow[r] \arrow[d, "\mathbf{x}_0" ', "\ell" '] & t, \mathbf{x} + \Delta\mathbf{x} + t\mathbf{u}_0, \mathbf{u} + \mathbf{u}_0 \arrow[d, "\mathbf{x}_0", "\ell"] \\
\tau, \mathbf{X}, \mathbf{U} \arrow[r, "\mathbf{x}_0 \mapsto \tilde{\mathbf{x}}_0"] & \tilde{\tau}, \mathbf{X}, \tilde{\mathbf{U}}
\end{tikzcd}$$

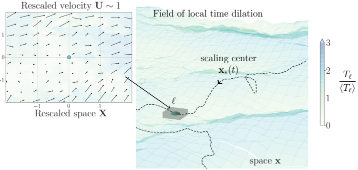
- **Physics:** $\exists$ Inertial range for hidden Navier-Stokes

- **Math:** $\exists \mathbb{P}_{HS}(dW) = \lim_{\tau\to\infty} \lim_{\ell\to 0} \lim_{\nu\to 0} \frac{1}{\tau}\int_0^\tau \mathbb{1}_{\phi[\mathbf{u}]\in dW}$

Generalized multipliers $W = \Phi_\lambda[\mathbf{U}]$



1. Dynamical rescaling

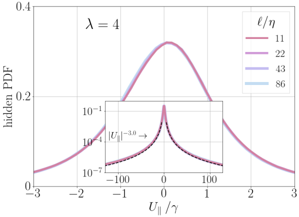


$$\partial_\tau \mathbf{U} + \Lambda_{\mathbf{U}} \left[\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P \right] = 0$$

2. Fusing old into hidden symmetries

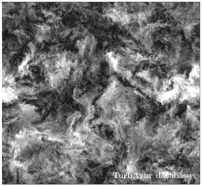
$$\begin{tikzcd}
t, \mathbf{x}, \mathbf{u} \arrow[r] \arrow[d, "\mathbf{x}_0 \\ \ell"'] & \lambda^{1-h} t, \lambda \mathbf{x}, \lambda^h \mathbf{u} \arrow[d, "\lambda \mathbf{x}_0 \\ \ell"] \\
\tau, \mathbf{X}, \mathbf{U} \arrow[r, "\ell \mapsto \ell/\lambda"] & \tau', \mathbf{X}, \mathbf{U}'
\end{tikzcd}$$

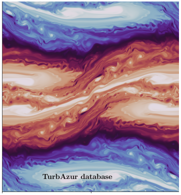
3. Statistical hidden universality

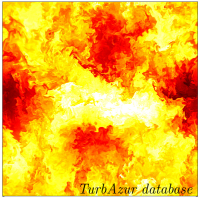


1. Intermittency
2. Conservation laws (Zero-modes)
3. (Hidden) Symmetries
4. **Hidden fluid mechanics**

Scope of Hidden Symmetry





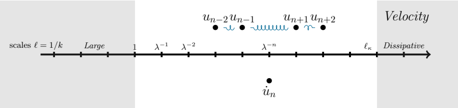


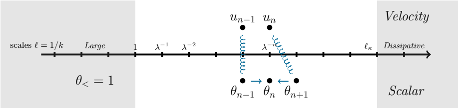
('22)

IN PROGRESS

Non-linear

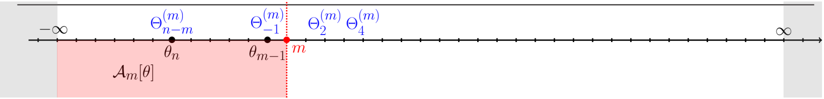
Linear





Mailybaev ('21, '22,'23)

('24)



1. Hidden KWB

$$d\Theta = \circ\Lambda_\Theta\left[\mathcal{N}\left[\Theta, dW\right]\right],$$

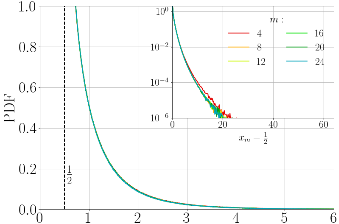
with

$$\Lambda_\Theta[V] = V - \Theta \sum_{J\leq 0} \alpha^{-J}\Theta_J V_J.$$

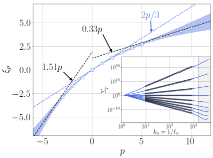
2. Fusing symmetries

$$\begin{tikzcd}
t, \theta \arrow[r, "n \mapsto n+1"] \arrow[d, "m"'] & \lambda^{4/3}t,\ \lambda^h\theta \arrow[d, "m+1"] \\
\tau, \Theta^{(m)} \arrow[r, "m \mapsto m+1"] & \lambda^{4/3}\tau, \Theta^{(m+1)}
\end{tikzcd}$$

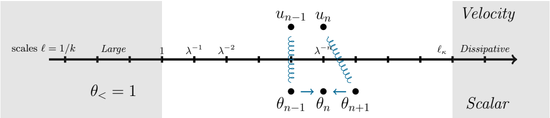
3. Scale invariance of multipliers

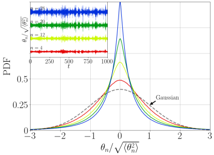


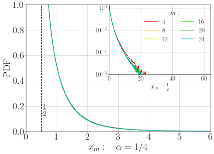
(4. Extraction of scaling exponents)



From zero-mode intermittency to hidden symmetry







Zero-modes	**Hidden Symmetry**
$\mathcal{M}_{2n}\ \langle\theta\theta\cdots\theta\rangle = 0$	$d\Theta = \circ\Lambda_\Theta\left[\mathcal{N}\left[\Theta, dW\right]\right],$
Conservation laws Even-order exponents	Fusing of Symmetries Multipliers universality
Non-linear settings?	Computation?

ArXiv: https://arxiv.org/abs/2402.04198