Questioning turbulence with surface quasi-geostrophy

Simon Thalabard

joint work with Nicolas Valade and Jéremie Bec

Valade, PhD Thesis '25
JFM '25 arXiv: 2503.16294
AHP '22, arXiv: 2210.12366

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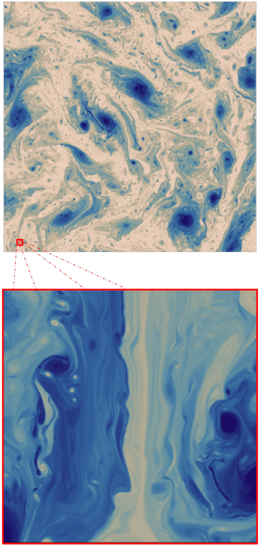
1. **Surface Quasi-Geostrophy**



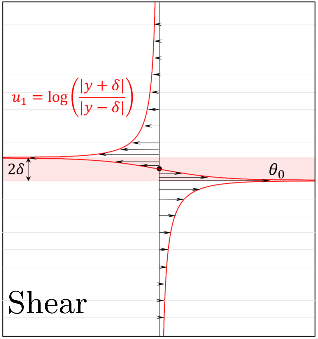
2D active transport   Blumen ’78, Juckes ’94, Held ’95

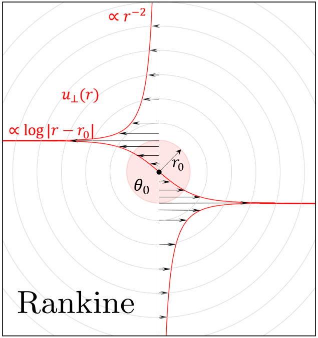
$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = \text{Forcing} + \text{Dissipation}$ — surface temperature

$\mathbf{u}(x,y) := -\nabla^{\perp}\psi,$ — horizontal advection; divergence-free

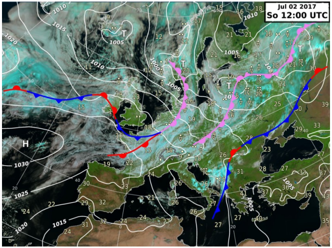
$\psi := (-\Delta)^{-\frac{1}{2}}\theta,$ — **Unconventional** stream function

Prototypical flows

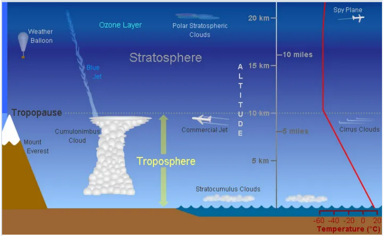




Front dynamics: Strong stratification $\sim$ Rotation



From MeteoSwiss

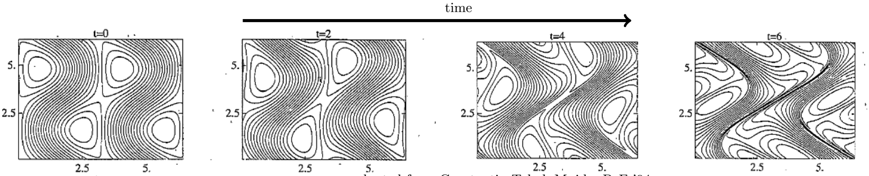


This diagram shows some of the features of the troposphere.
UCAR/Randy Russell

A few references

- *Derivations:* Blumen, JAS '78; Juckes, JAS '94; Held et al, JFM '95
- *Tropopause:* Tulloc & Smith, JAS '07, JAS '08
- *Upper Ocean:* Lapeyre & Klein, JPO '06; LaCasce & Mahadevan, JMR '06; Fresney-al, JGR '18...
- *Review:* Lapeyre, Fluids '17
- *Next-order SQG*: Hakim-al, JAS '02, Maalouly, PRF '24,...

A 2D model (likely) generating singularity

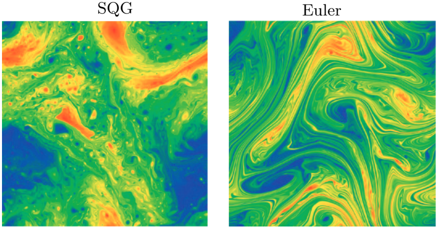


adapted from Constantin-Tabak-Majda, PoF '94

A few references

- *Deterministic Blow-up:* Constantin et al, NL '94; Majda & Tabak PhysD '98; Cordoba '97; Chae-al, '10; Constantin-al JSC '12,...., Cordoba-al, 17, Córdoba-Martinez-Zoroa '25, Ferreira-Guimares '25
- *Non-uniqueness:* Buckmaster, Shkoller & Vicol, CPAM '16;
- *Regularization by noise:* Flandoli -al '21; Jiao-Luo '24; Galeati et al '24...

Rough nonlinear 2D scalar transport



from Watanabe & Iwayama, JPSJ '04

Analogies with canonical flows:
2D and 3D NS, Passive scalar, Burgers.

A few references

- *Cascades and Scaling:* Sukhathme & Pierrehumbert, Chaos '02; Capet et al, JFM '08; Lapeyre, '17; Smith et al, JFM '22; Valadao et al, '25; Valade et al, JFM '25
- *Generalized SQG:* Pierrehumbert et al Chaos '95; Watanabe & Iwayama JPSP '04; Smith et al, JFM '22
- *Predictability:* Rotunno & Snyder '07; Valade et al AHP '23; Valadao et al, '25
- *Transport:* Celani et al, PRL '02, NJP '05, Foussard-al, JFM '17; Maalouly et al PRF '24

Structural analogies

- **2D:** Two quadratic conservations laws

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^2} \psi\,\theta \quad \text{(Hamiltonian)} \quad \mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^2} \theta^2 = \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{u}|^2 \quad \text{(energy)}$$

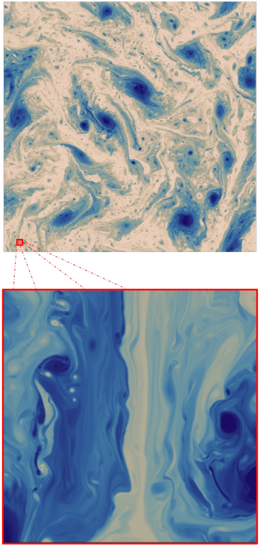
- **Burgers:** $[\mathbf{u}] = [\theta]$
- **3D:** Vortex stretching

$$\partial_t \omega + \mathbf{u}\cdot\boldsymbol{\nabla}\omega = -\boldsymbol{\nabla}\cdot\boldsymbol{a} + F + D, \quad \boldsymbol{a} := (-\Delta)^{1/2}[\mathbf{u}\,\theta] - \mathbf{u}\,\omega$$

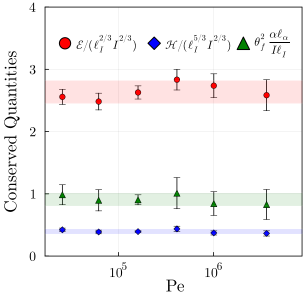
SQG turbulence: Analogous to 3D HIT... yet unconventional

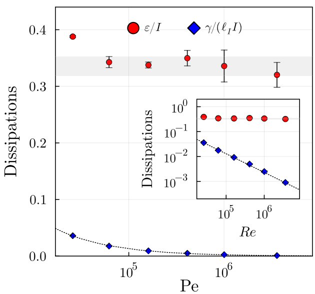
Steady-state	**Decaying**
Unconventional intermittency	Unconventional predictability

1. Surface Quasi-Geostrophy
2. **A refined view on the direct cascade**



$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = F + \nu\Delta\theta, \quad \hat{F}_{\mathbf{k}} = \sqrt{\frac{2\mathcal{I}}{c_d\pi k_\mathrm{f}^2}}\,\mathrm{e}^{-\frac{|\mathbf{k}|^2}{2k_\mathrm{f}^2}}\,\hat{\eta}_{\mathbf{k}} - \alpha\,\hat{\theta}_{\mathbf{k}}\,\mathbf{1}_{|\mathbf{k}|\le k_\mathrm{f}}.$$





Prototypical K41 turbulence

i. Steady State. $Pe = \frac{\theta_f}{\nu k_{\mathcal{I}}}$

ii. Finite variance

iii. Dissipative anomaly

iv. Power-laws $E(k) \propto \varepsilon^{2/3} k^{-5/3}$

Specificities

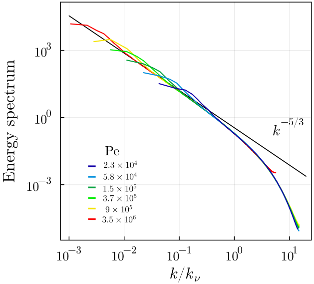
i. Dissipative anomaly $\varepsilon < \mathcal{I}$

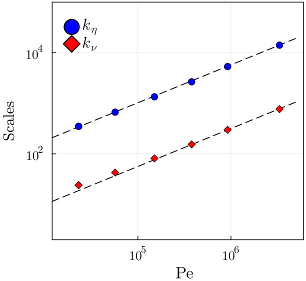
ii. Dissipative scale:

$$k_\nu = \frac{\langle|\nabla\theta|^2\rangle}{\langle\nabla\theta\cdot\nabla\psi\rangle} > k_\eta.$$

iii. Structures

$$\partial_t\theta + \mathbf{u}\cdot\boldsymbol{\nabla}\theta = F + \nu\Delta\theta, \quad \hat{F}_{\mathbf{k}} = \sqrt{\frac{2\mathcal{I}}{c_d\pi k_\mathrm{f}^2}}\,\mathrm{e}^{-\frac{|\mathbf{k}|^2}{2k_\mathrm{f}^2}}\,\hat{\eta}_{\mathbf{k}} - \alpha\,\hat{\theta}_{\mathbf{k}}\,\mathbf{1}_{|\mathbf{k}|\le k_\mathrm{f}}.$$





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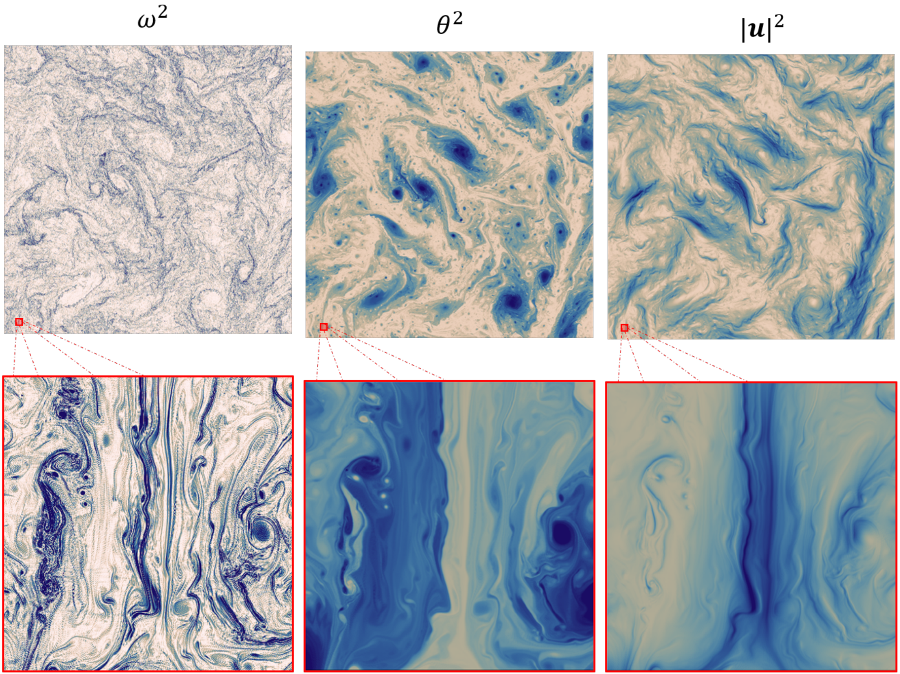
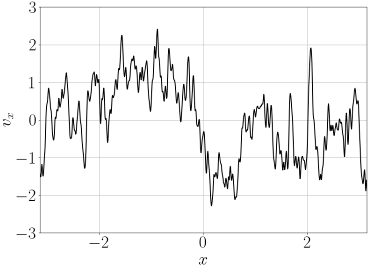
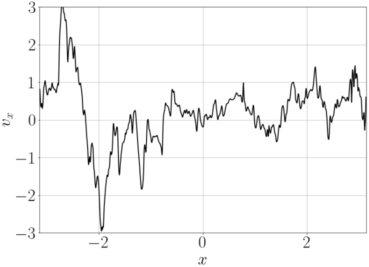

ω^2
θ^2
$|\boldsymbol{u}|^2$

Intermezzo: 3D Navier-Stokes vs Gaussian turbulence





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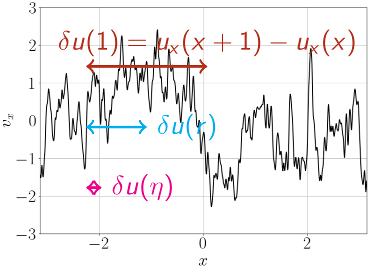
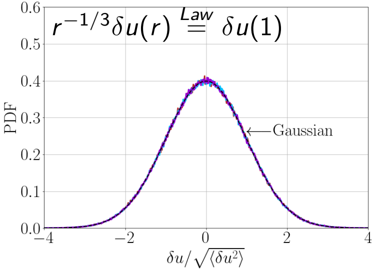
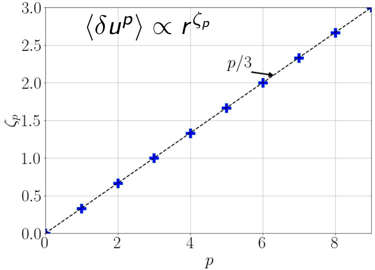
Intermezzo: Gaussian turbulence



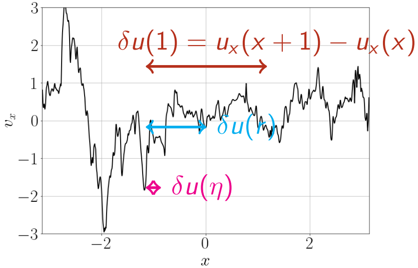


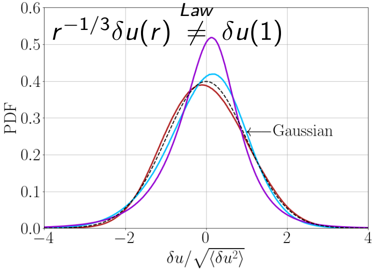
Scale invariance

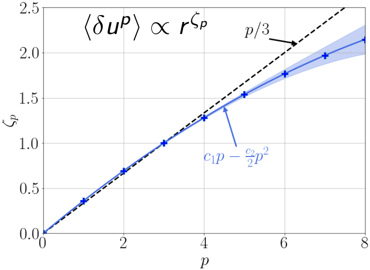


Monofractal scaling

Intermezzo: Navier-Stokes turbulence



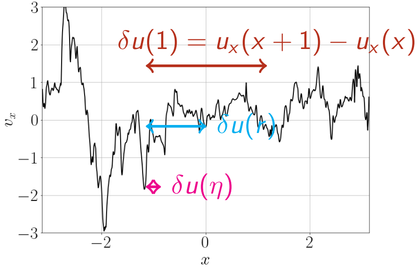


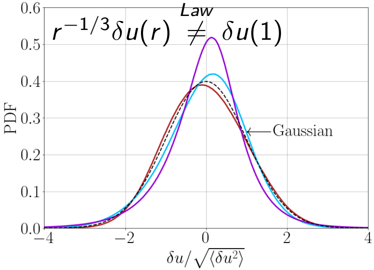


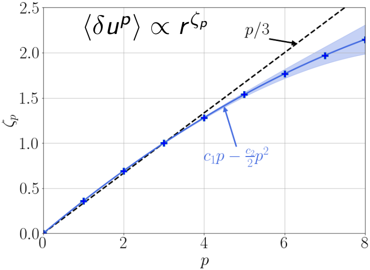
No obvious scale invariance

Multifractal scaling

Intermittency :
Deviations from Gaussian turbulence







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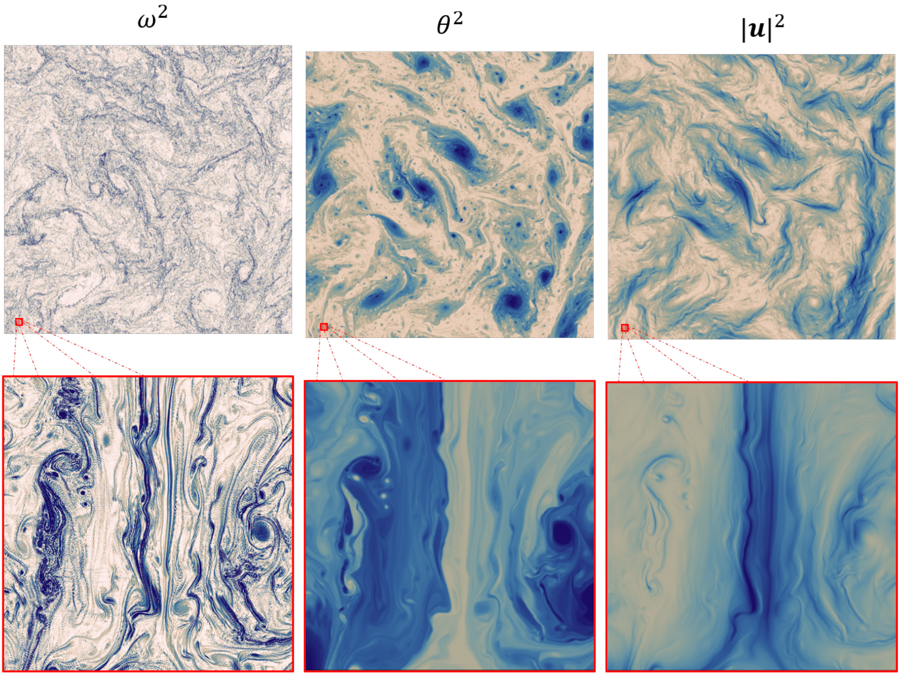
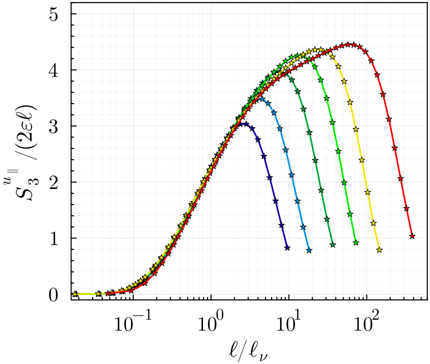
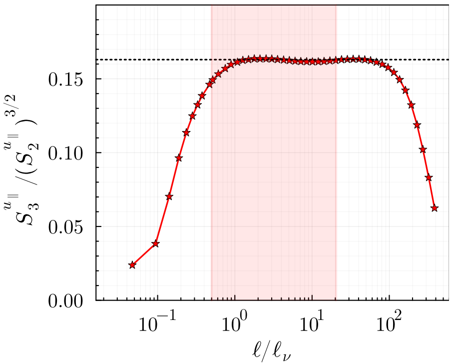

ω^2
θ^2
$|\boldsymbol{u}|^2$

SQG intermittency: (i) Skewness anomaly





Anomalies at the level of the longitudinal increments $\delta u_\parallel$

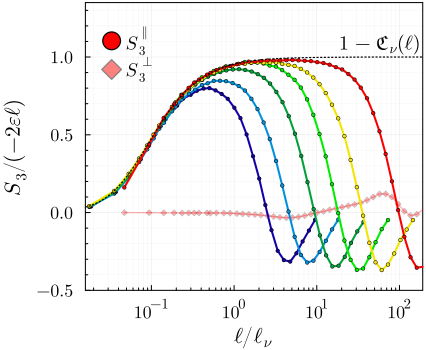
✓ Deviations from K41 scaling

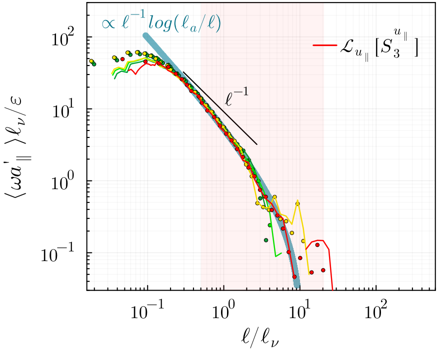
$$\left\langle \delta u_\parallel^3 \right\rangle \not\propto \epsilon\ell$$

! Positive skewness

Capet, Klein, Hua , Lapeyre & McWilliams
JFM '08

SQG intermittency: (i) Skewness anomaly





! cascade $\perp\!\!\!\perp$ skewness

! Log-corrections

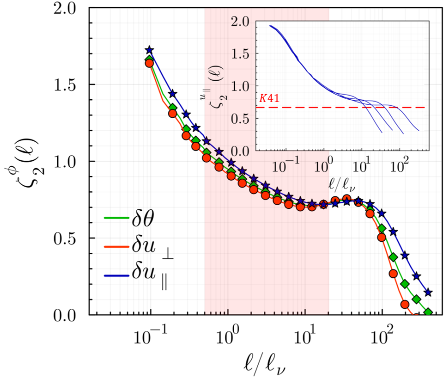
- Yaglom's law: $\left\langle \delta U (\delta\Theta)^2 \right\rangle = -2\varepsilon\xi$
- Kinematic relation Bernard, PRE '99

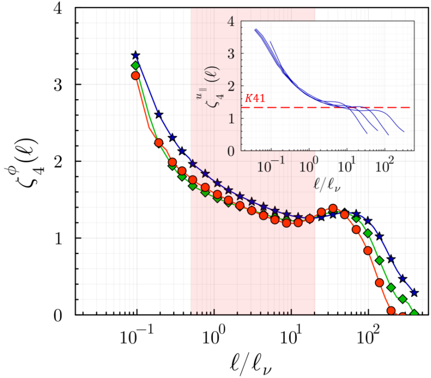
$$\partial_{\xi\xi}\langle(\delta U)^3\rangle = 3\partial_{\xi\xi^*}\langle \delta U(\delta\Theta)^2\rangle - 3\langle A(0)\Omega(\xi)\rangle$$

$$\boldsymbol{a} := (-\Delta)^{1/2}[\mathbf{u}\theta] - \mathbf{u}\omega$$

. ("vortex stretching")

Scaling exponents: $\zeta_p := \dfrac{\mathrm{d}\log\langle|\delta\phi|^p\rangle}{d\log\ell}$





- **!** Single-fields: No power-law
- **!** Mixed correlators: Power-law scaling for fluxes of higher-order conservation laws

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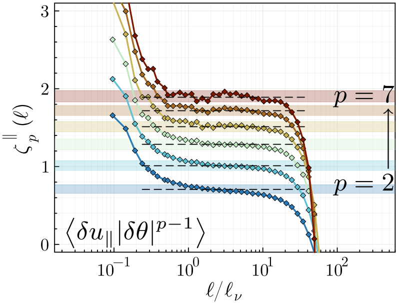
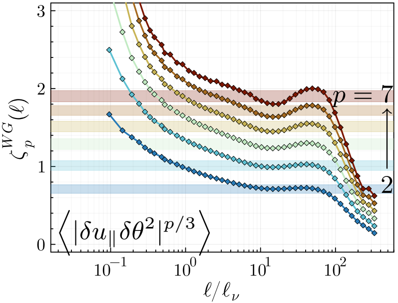
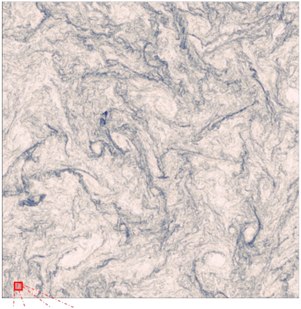




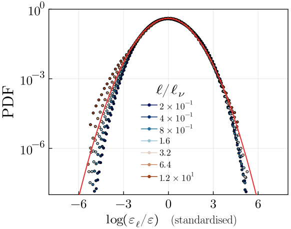
- **!** Single-fields: No power-law
- **!** Mixed correlators: Power-law scaling for fluxes of higher-order conservation laws

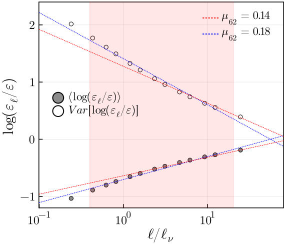
$|\nabla\theta|^2$



Local average of the dissipation field

$$\varepsilon_\ell = \nu \left\langle |\nabla\theta|^2 \right\rangle_\ell$$





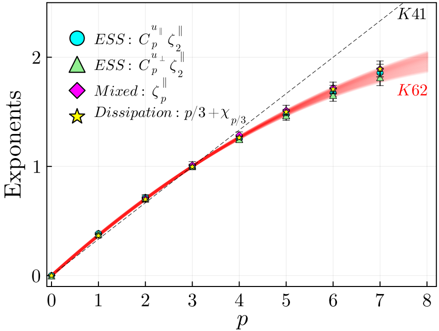
✓ Approximate log-normality

$$\langle \varepsilon_\ell^q \rangle \sim \ell^{\chi_q}$$

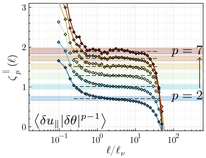
$$\chi_q \approx \mu_{62}\, q\, (1-q)$$

$$\mu_{62} = 0.16 \pm 0.02.$$

3D HIT, *e.g.* Iyer et al, '15, Chevillard-al '19.

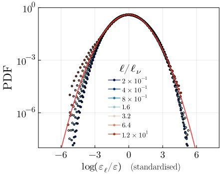


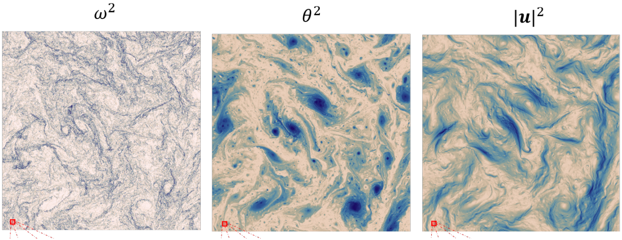
! Refined similarity



$$\langle \delta \mathbf{u}_\parallel |\delta\theta|^{\mathbf{p}-1}\rangle \sim \ell^{\mathbf{p}/3}\langle \varepsilon_\ell^{\mathbf{p}/3}\rangle,$$

$$\zeta_p^\parallel = \chi_{p/3}(\ell) + p/3$$

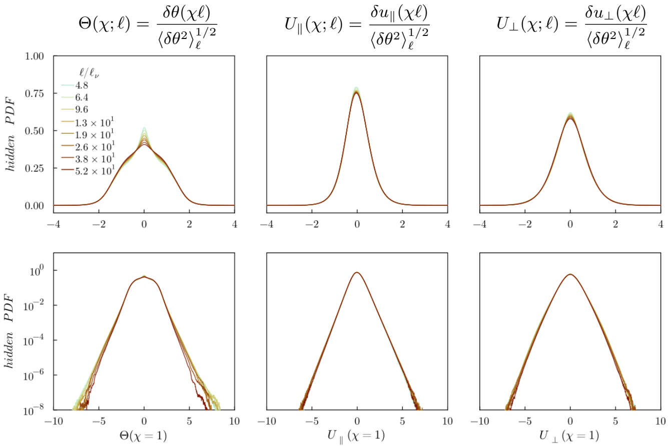




- ✓ K41 phenomenology
- ✓ Deviations from K41 scaling
- ✓ Refined similarity
- ✓ Approximate log-normality

- ! Power-law scaling restricted to selected correlators $(\delta u_{\|}|\delta\theta|^{p-1})$
- ! Single-fields:
  - ! No power-law (log-corrections)
  - ! Positive skewness
  - ! cascade $\perp\!\!\!\perp$ skewness

Multiscaling shaped by conservation laws rather than symmetries ?

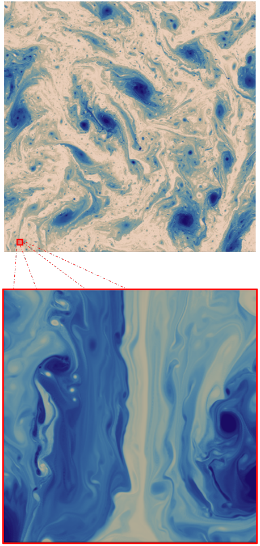


Valade, PhD Thesis '25

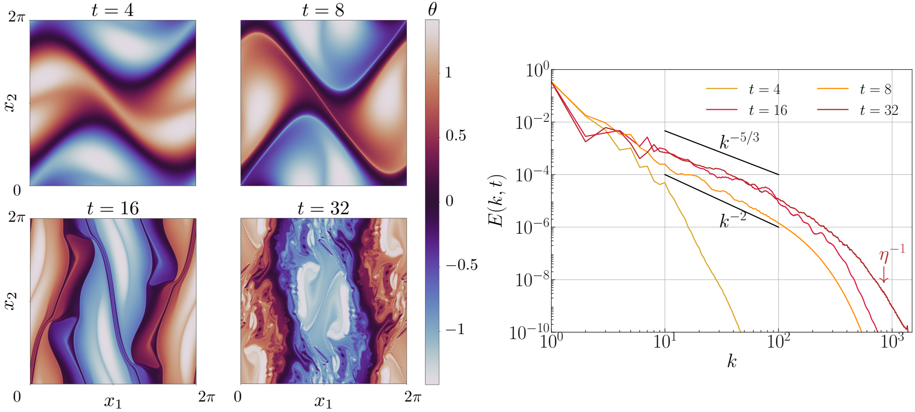
- Scale invariance of multipliers?

  Kolmogorov '62; Chen et al, PRL '03; Eyink et al JSP '03
- Multiplicative origin of multiscaling? (Hidden symmetry) Mailybaev '22^2, '23 , ST & Mailybaev '22,'24, Magacho et al '25, Calascibetta et al '25

1. Surface Quasi-Geostrophy
2. A refined view on the direct cascade
3. **Predictability of deterministic SQG**

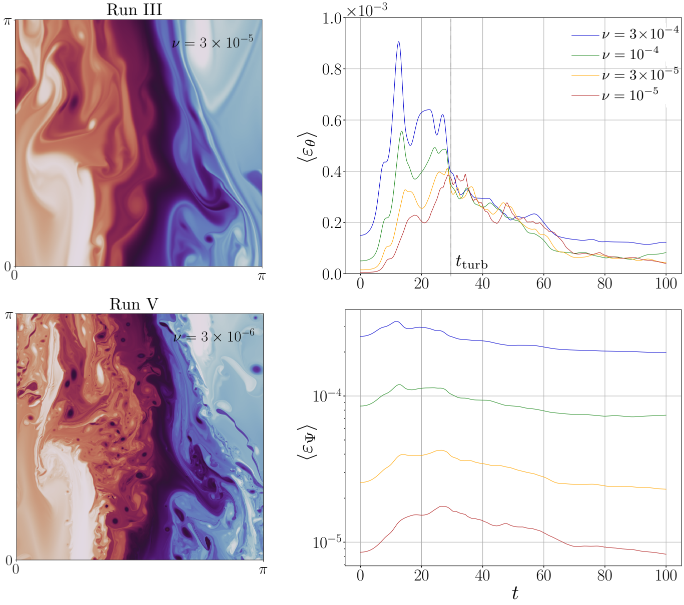


$\theta(\mathbf{x}, t=0) = \cos x_2 - \sin x_1 \sin x_2$ Constantin, Majda & Tabak, Nonlinearity '94



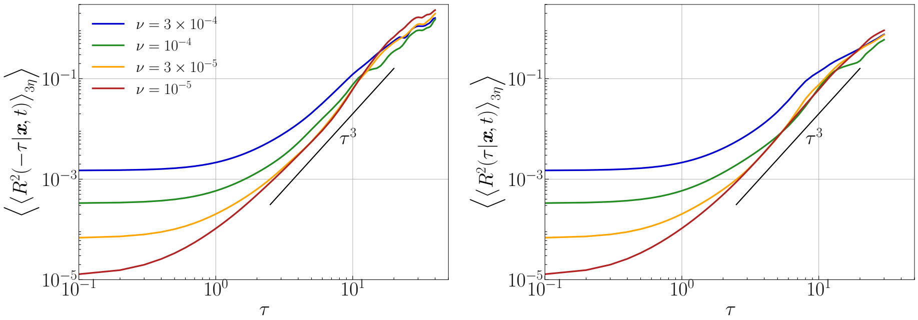
Finite-time emergence of a <u>rough</u> regime corresponding to SQG turbulence.

Anomalous dissipation



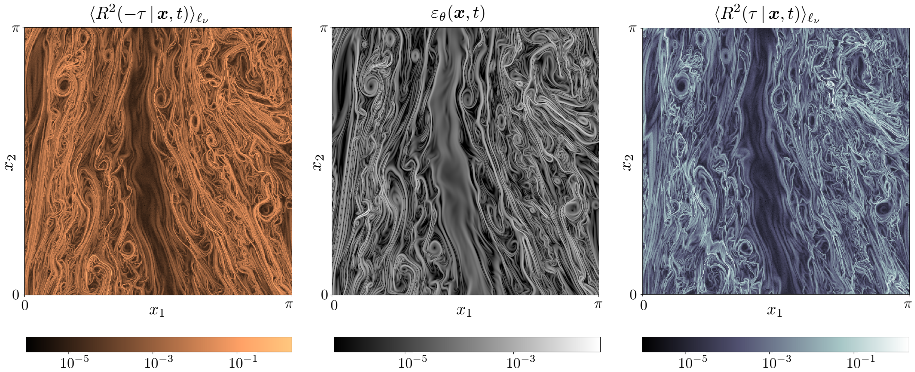
Lagrangian predictability: (i)K41 $\implies$ Richardson super-diffusion

$$\delta u_\parallel \propto \varepsilon^{1/3} r^{1/3} \Rightarrow R(t) \propto \varepsilon^{1/2} t^{3/2}$$



akin to 3D direct cascade and 2D inverse cascade

Lagrangian predictability:
(ii) Eulerian dissipation $\Longrightarrow$ Non-deterministic transport



Stochastic representation of transport through Ito formulas

Revuz-Yor, Gawedzki '06. Eyink-Drivas '16

$$\underbrace{\mathcal{E}(0)-\mathcal{E}(t)}_{\text{Eulerian Dissipation}} = \frac{1}{8\pi^2}\iint \underbrace{p^\nu(\mathbf{y},0\,|\,\mathbf{x},t;\theta_0)}_{\text{Lagrangian fluctuation}} \; (\theta(\mathbf{x},t)-\theta_0(\mathbf{y}))^2 d\mathbf{x}\, d\mathbf{y}.$$

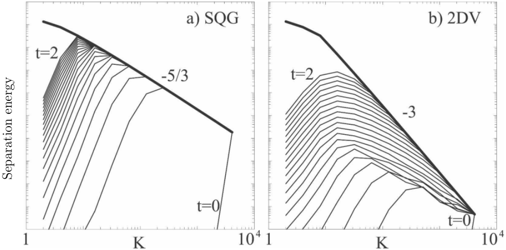
aka: Production of uncertainty/Predictability Horizon/ Spontaneous stochasticity

Inverse error cascade : $\delta u(\ell) \sim \ell^h$, $T_{2\ell \leftarrow \ell} \propto \ell^{1-h}$

$$T_{1 \leftarrow 0} \propto \sum_{n \geq 0} T_{2^{-n} \leftarrow 2^{-n-1}}.$$

$h = 1/3$

$T_{1 \leftarrow 0} < \infty$



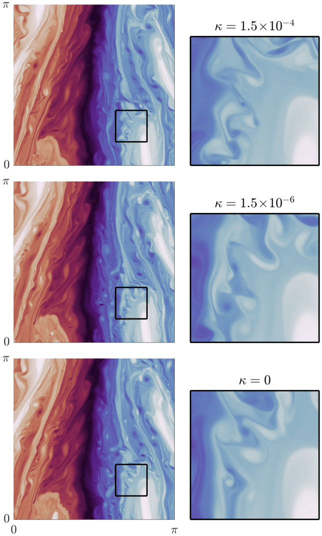
From Rotunno & Snyder, JAS '07

$h = 1$

$T_{1 \leftarrow 0} = \infty$

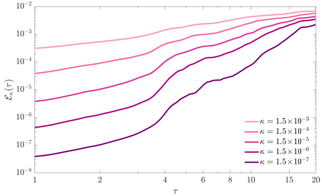
A few references:

- *Theory:* Lorenz JAS '69; Palmer et al NL '14; Eyink & Peng, NL' 25;
- *Simplified models:* Leith & Kraichnan JAS '72; Métais & Lesieur JAS '86; Rotunno & Snyder JAS '08; Mailybaev, since '12; ; Bandak et al, PRL '24; Ortiz et al '25;
- *DNS:* Boffetta et al, PoF '96 ; Boffetta & Musacchio, PoF '01, PRL' 17; Thalabard et al, CP '20; Ge et al, JFM '23; Valadao et al, JFM '25

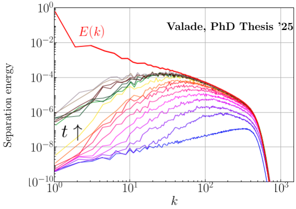


Separation energy:

$$\mathcal{E}_\kappa(\tau) = \frac{1}{2}\left\langle \left(\theta^{(\kappa)}(\cdot, t+\tau) - \theta(\cdot, t+\tau)\right)^2 \right\rangle$$



! No explosive growth



! Error cascade blocked in the inertial range

! Large-scale predictability despite small-scale roughness

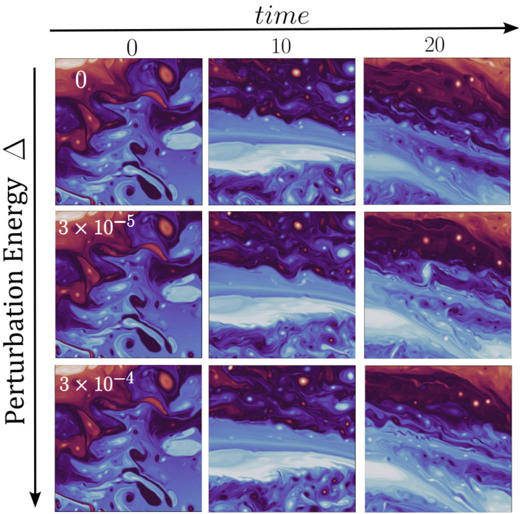


Figure 5.14: Snapshots of the hyper-viscous temperature fields $\theta^{(\Delta)}(\boldsymbol{x}, t+\tau)$, perturbed at time $t = 30$ with white noise of energy Δ. Only $1/20th$ of the full domain is shown.

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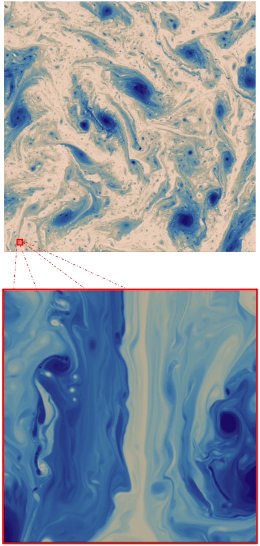
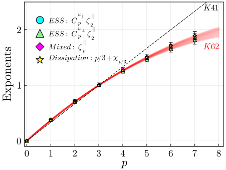
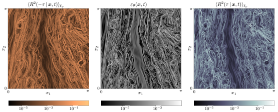
1. Surface Quasi-Geostrophy
2. A refined view on the direct cascade
3. Predictability of deterministic SQG
4. **Concluding remarks**



A rough 2D active scalar à la Kolmogorov...

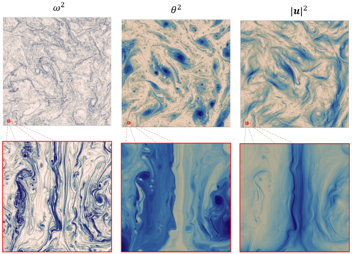
- Kolmogorov scaling
- Anomalies
  - Scaling (intermittency)
  - Dissipation
  - Lagrangian flow





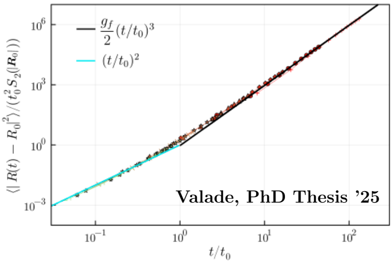
...but with fundamental specificities

- Entanglement of structures
- Conservation laws
- Positive skewness but direct cascade!
- Non-universality

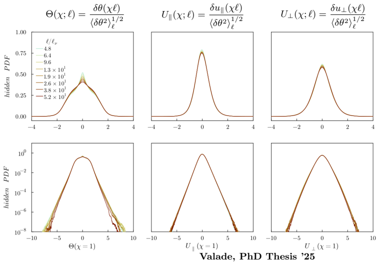


Valade, PhD Thesis '25
JFM '25, arXiv 2503.16294
AHP '22, arXiv 2210.12366

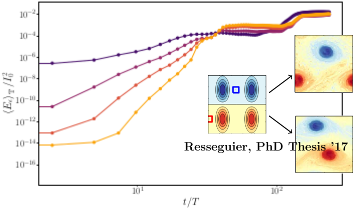
Lagrangian predictability



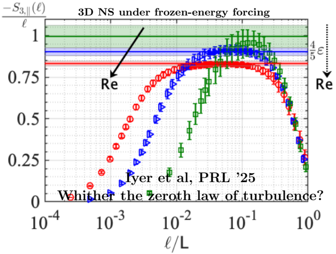
Multipliers



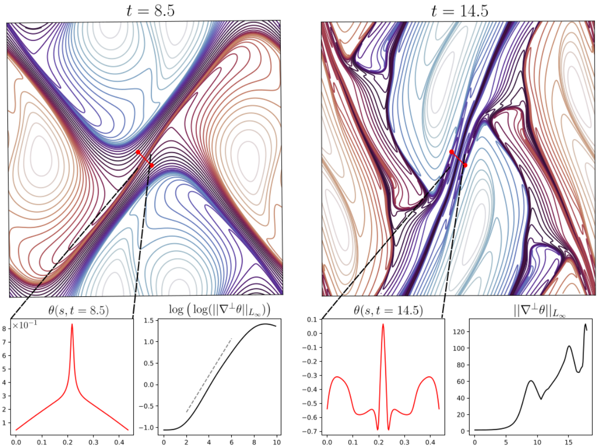
Eulerian predictability
(with Barlet, Li & Mémin)



Universality



Is $t_{turb} < \infty$?



Valade, PhD Thesis '25

- Finite-time singularity: $\int_0^{t_*} \|\nabla^\perp\theta\|_\infty = \infty$ Constantin et al '97, Cordoba '97
- Data suggests: $\|\nabla^\perp\theta\|_\infty \propto e^{e^t}$ Ohkitani & Yamada, PoF '97, Cordoba '98
  (Implying practical blowup at $t \simeq 7$ using floating precision.)